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Data inversion in MCDM problems: nonlinear 1/a and linear ReS inversion

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Abstract. A comprehensive analysis of the procedures for consistent normalization/inversion of benefit and cost attributes for multi-criteria decision making (MCDM) and multivariate classification problems is performed. This study demonstrates that the commonly used $1/a$ transformation for cost attribute inversion introduces structural inconsistencies in normalized data. Nonlinear data inversion does not have a reasonable interpretation of values and leads to a violation of mutual distances in the original data. The measurement scales of various attributes are not consistent and there is a shift in the domains of normalized values. Elimination of these problems is achieved by using the linear inversion Reverse Sorting algorithm (ReS). The ReS algorithm offers more consistent, linear, and interpretable results for handling cost attributes in MCDM. The ReS algorithm is a linear transformation and preserves the original information about the object: dispositions of attribute values, preserves the relative positions of the domains of different attributes and can be applied to both the original and normalized data sets. The ReS algorithm eliminates all the shortcomings of nonlinear inversion and is recommended for inversion of values when coordinating the optimization goals of a multi-criteria problem, as well as in the weighing methods based on information contained in the decision matrix.

Keywords: MCDM, normalization of multivariate data, non-linear inversion, Reverse Sorting algorithm (ReS), scale coordination

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1. Introduction

The standard problem of multi-criteria choice on a discrete set of alternatives (MCDM) is formed as follows [4, 12]: for an available set of alternatives $A_i, (i = 1, \ldots, m)$ the attributes a_{ij} (decision matrix) are determined in the context of the selected evaluation criteria $C_j, (j = 1, \ldots, n)$. Then a decision model is formed. The most widespread models are based on Value Measurement methods, which include the method of assessing the significance of criteria (W), the method of normalizing the decision matrix ($Norm$) and the method of aggregation (F) of various attributes of each alternative into an indicator of its effectiveness Q_i:

$$Q_i = F(x_{ij}, w_j), \quad \forall i = 1, \ldots, m. \tag{1}$$

Where, F is one of the methods or functions (including multi-step) of data aggregation. F transforms the normalized values of the attributes $x_{ij} = Norm(a_{ij})$, taking into account the significance of the criteria w_j, to the numerical value Q_i — the rating of the alternative (or the

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performance indicator, or the assessment score). For example, one of the simplest and most widely used aggregation methods is a weighted sum of the normalized values of the attributes of each alternative ($Q_i = \sum_{j=1}^{n} w_j x_{ij}$).

The choice of the trio of methods (F, W, $Norm$) is not formalized and is carried out on the basis of some general principles [8].

Additive (or multiplicative) aggregation functions F assume consistency in the direction of improvement of each attribute. There are three possible types of goals for different criteria: larger the best (LTB), smaller the best (STB), and target the best (TTB). Aggregation involves matching the goals of the criteria if the goal type of different criteria is different. For Value Measurement Methods, matching the goals is achieved by data inversion. We define data inversion as a rearrangement of the sorted list in reverse order. With a common LTB goal, the cost criteria are inverted, and with an STB goal, the benefit criteria are inverted. For criteria with a specified target value, with a common LTB goal, the inversion is performed for all values greater than the target, and with an STB goal, the inversion is performed for all values less than the target [8]. Inversion methods are also multivariate. Following the review by Jahan (2015) [5], and other studies, the understanding of this fact is exaggerated and the term inversion is not used, but separate coordinated normalization of benefit criteria (LTB type) and cost criteria (STB type) is applied. In this case, two separate formulas are used. For example, for Max-method of normalization, the formula is:

$$x_{ij} = \frac{a_{ij}}{a_j^{max}}, \forall j, \quad \text{then} \quad x_{ij} := 1 - x_{ij}, \text{ for } \forall j \in C_j^{-} \text{— cost criteria.} \tag{2}$$

That is first normalization of all attributes is performed, and then a linear inversion of the form (1–x) is performed for cost attributes (or benefits for the general goal of STB).

In formula (2) and further, the assignment operator ":=" is used, which is interpreted as follows: the variable on the left is assigned the result of the operation performed on the right. This will allow one variable to be operated upon when performing the normalization/inversion operation.

Normalization of multidimensional data is not only bringing data to a dimensionless form, but also coordinating normalization scales. This is a necessary element of transforming multidimensional data. The requirement is that none of the data sets receive priority over the others at the normalization stage. In fact, this is impossible [8], since each of the features uses its own normalization scale. Considering that inversion is an integral part of multidimensional data normalization, the same requirements are imposed on the inversion procedure as on normalization.

In this paper, we will focus on the analysis of the procedure for the consistent normalization of the benefit and cost criteria $x_{ij} = Norm(a_{ij})$, or more precisely, on the inversion of the objective depending on the overall objective of multi-criteria optimization. Normalization/inversion is relevant for MCDM problems for Value Measurement models and for Goal or Reference Level models. In the second case, inversion is not required. Therefore, this article discusses various data inversion methods as applied to solving MCDM problems using Value Measurement Methods. This inversion is often used as a component of such models as, for example, Weighted Sum Model (WSM), Weighted Product Model (WPM), Weighted Aggregated Sum Product Assessment (WASPAS) [15], COmplex PRoportional ASsessment (COPRAS) [14], Additive Ratio Assessment (ARAS) [16], COmbinative Distance-based ASsessment (CODAS) [3], Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [12], etc. Exactly the same approach is used in objective methods for assessing the weight of criteria based on the information contained in the decision matrix: Entropy weight method (EWM) [13], MEthod based on the Removal Effects of Criteria (MEREC) [6], Simultaneous Evaluation of Criteria and Alternatives (SECA) [2], method of Criterion Impact LOSs (CILOS) [1], etc.

It is clear that the monotonic a strictly decreasing function $1/a$ will preserve the order of the values in the list and invert the data, so that larger values become smaller and vice versa. In this study, based on the analysis and simple examples, we will show that such an inversion is not correct. A nonlinear transformation changes the data, or rather changes the distances between the values. This means that the normalized data is fundamentally distorted. Thousands of studies in MCDM using nonlinear inversion of cost attributes based on a transformation of the type $1/a$ should be recognized as inaccurate. As an alternative, the paper proposes a linear method, Reverse Sorting (ReS), an algorithm that eliminates all the problems of nonlinear transformation. The ReS algorithm is a linear transformation and preserves the original information about the object: dispositions of attribute values, preserves the relative positions of the domains of different attributes and can be applied to both the original and normalized data sets.

Although the basic publication of the linear inversion of ReS in the journal IJITDM [7] was in 2020, very few studies use this method. Meanwhile, this is the simplest and most universal inversion for both univariate and multivariate data. In this paper, the authors aim to show that the only correct solution is to use the ReS inversion, which satisfies all the necessary requirements for normalizing multidimensional data.

2. Methods: data transformation in multi-criteria optimization problems

In problems with a multidimensional data set, as a rule, preliminary transformation of the values of all features is required so that they fall into intervals of comparable size. This procedure of bringing values measured in different scales to a conditionally common scale is defined as normalization. Multidimensional normalization transforms data independently for each attribute.

2.1. Linear normalization

The general formula for linear normalization is:

$$x_{ij} = \frac{a_{ij} - a_j^*}{k_j}, \quad i = 1, ..., m, j = 1, ...n. \tag{3}$$

The shift coefficients a_j^* and stretching-compression coefficients k_j are different for each criterion. This is due to the fact that the attributes of objects and the ranges of their values can differ greatly from each other. Therefore, each of the attributes uses its own normalization scale and the normalized values depend on the measurement scale and on the range natural values of the attributes ($r_j = range_i(a_{ij}) = a_j^{max} - a_j^{min}$).

2.2. Invariance of attribute value dispositions under linear transformations

We define the disposition of the values a_{ij} of the jth attribute for alternatives A_p and A_q as:

$$\Delta a_{pq}^{(j)} = \frac{|a_{pj} - a_{qj}|}{a_j^{max} - a_j^{min}}. \tag{4}$$

Without taking into account the sign, the disposition is the relative distance between two attribute values of alternatives A_p and A_q. It is easy to show [8, 9] that the disposition of the values of the jth attribute for the alternatives A_p and A_q is an invariant with respect to linear transformations $x = L(a)$:

$$\Delta a_{pq}^{(j)} = \frac{|a_{pj} - a_{qj}|}{a_j^{max} - a_j^{min}} = \frac{|x_{pj} - x_{qj}|}{x_j^{max} - x_j^{min}} = \Delta x_{pq}^{(j)}, \quad \forall p, q = 1, ..., m, \quad \forall j. \tag{5}$$

The converse is also true: if the dispositions of two data sets are identical, then they are linearly transformed into each other.

The interpretation of this property is that relative distances between data are preserved under linear transformations. For a sorted list, property (5) has the following form:

$$\Delta a_i^{(j)} = \frac{a_{i+1,j} - a_{ij}}{a_{mj} - a_{1j}} = \frac{x_{i+1,j} - x_{ij}}{x_{mj} - x_{1j}} = \Delta x_i^{(j)}, \;\; i = 1, ..., m-1, \;\; \forall j. \tag{6}$$

Obviously, two such lists are ordered identically and differ in scale and position.

2.3. Invariance of dispositions of rating under linear transformations of data

We apply a linear transformation of the form:

$$u_{ij} = x_{ij}/k + h_j, \tag{7}$$

to the normalized values. Here, the compression coefficient k is a fixed constant for $\forall j$, and the bias h_j can be different for different attributes. That is, all attributes will be subjected to the same compression or expansion, but the shift of the domains of different attributes may be different. In the study [9], the invariance of rating dispositions was proven for the case of linear (WSM) or homogeneous (TOPSIS) aggregation functions F, according to formulas (1) under otherwise identical conditions.

Let us explain this important property.

Let us apply the MCDM rank model and determine the ratings of alternatives $Q_i^{(1)} = F(x_{ij}, w_j)$ in accordance with (1) for the normalized decision matrix x_{ij}, and determine the ratings of alternatives $Q_i^{(2)} = F(u_{ij}, w_j)$ for the transformed decision matrix u_{ij}. As the aggregation function F, we use a linear function (WSM and similar) or a homogeneous function (TOPSIS and similar). In both cases, we use the same aggregation functions and the same weighting coefficients. Invariance of the rating dispositions means that

$$\Delta Q_i^{(1)} = \Delta Q_i^{(2)}, \;\; \forall i = 1, \ldots, m-1. \tag{8}$$

From the equality of dispositions it follows that the ranks R_i of the alternatives are the same in both cases

$$R_i^{(1)} = R_i^{(2)}, \;\; \forall i = 1, \ldots, m. \tag{9}$$

Thus, a linear transformation of normalized data of the form $u_{ij} = x_{ij}/k + h_j$ does not change the dispositions of the rating list and does not change the ranks of the alternatives if a linear or homogeneous aggregation function is used.

2.4. Basic principles of multivariate data normalization

The basic principles of multivariate data normalization are set out in the recently published monograph [8]. Preservation of the information content of the data after transformation includes:

(i) Preservation of the information content of the data in the context of each attribute: For all linear normalization methods and linear transformations, the dispositions of natural values for each attribute are preserved according to property (5). The use of nonlinear transformations violates the structure of the original data and requires meaningful justification.

(ii) Alignment of normalized value scales: Each attribute has its own normalization scale, which is not aligned with the normalization scale of other attributes. This is due to the different range of values and different distribution of features in the domain. For multi-criteria problems, linear methods produce anisotropic scaling, when at least one of the scaling factors differs from the others. There is a deformation of the multidimensional feature cloud. This can lead to the priority of the contribution of individual attributes to the efficiency indicator of alternatives. The degree of difference depends on the decision matrix D (on the problem) and the choice of the data normalization method. Since it is impossible to reach a consensus on the alignment of normalized value scales at the normalization stage, it is necessary to minimize possible consequences when choosing normalization/inversion methods.

3. Data inversion procedures

3.1. Linear inversion of normalized values of the form (1-x)

One popular approach combines normalization and inversion. First, normalize all attributes, and then perform a linear inversion of the form $(1 - x_{ij})$ for the cost attributes. For example, for the Max method, the normalization/inversion is:

$$x_{ij} = \frac{a_{ij}}{a_j^{max}}, \ \ \forall i, j, \tag{10}$$

$$x_{ij} := 1 - x_{ij}, \text{ for } j \in C_j^- \text{ — cost criteria.} \tag{11}$$

In this case, if the normalized values x_{ij} are shifted to 1, as for the Max method, then $(1 - x_{ij})$ will be shifted to 0 and the contribution of cost criteria to the integral indicator will obviously be lower than the contribution of benefit criteria. However, since (11) is a linear transformation of the form (7) (k=-1, $h_j = 1$), then in accordance with property (8)-(9), this will not affect the ranking and does not change the dispositions of alternatives in the case of using the WSM or TOPSIS aggregation methods, including analogues.

Note that the above is also true for the inversion –x:

$$x_{ij} := -x_{ij}, \text{ for } j \in C_j^- \text{ — cost criteria.} \tag{12}$$

In this case, if the normalization transforms the values into [0, 1], then when using the inversion (12) $x_{ij} \in [-1, 0]$.

Inversions (11), (12) are obviously problematic when using nonlinear methods of aggregating normalized values of different attributes.

3.2. Nonlinear inversion of the form 1/a

Linear normalization procedures for benefit (C_j^+) and cost (C_j^-) criteria for the overall LTB goal are as follows:

$$x_{ij} = Norm(a_{ij}) = \frac{a_{ij} - a_j^*}{k_j}, \text{ for } j \in C_j^+, \quad x_{ij} = \frac{1/a_{ij}}{t_j}, \quad \text{for } j \in C_j^-. \tag{13}$$

1) Max/ iMax (inversion Max with non-linear inversion iMax)

$$x_{ij} = Max(a_{ij}) = \frac{a_{ij}}{k_j}, \quad a_j^* = 0, \ k_j = a_j^{max}, \text{ for } j \in C_j^+, \quad t_j = \frac{1}{a_j^{min}}, \text{ for } j \in C_j^-. \tag{14}$$

2) Sum/ iSum (inversion Sum with non-linear inversion iSum)

$$x_{ij} = Sum(a_{ij}) = \frac{a_{ij}}{k_j}, \ \ a_j^* = 0, \ k_j = \sum_{i=1}^{m} a_{ij}, \text{ for } j \in C_j^+, \quad t_j = \sum_{i=1}^{m} \frac{1}{a_{ij}}, \text{ for } j \in C_j^-. \tag{15}$$

3) Vec/ iVec (inversion Vec with non-linear inversion iVec)

$$x_{ij} = Vec(a_{ij}) = \frac{a_{ij}}{k_j}, \quad a_j^* = 0, k_j = \sqrt{\frac{1}{m}\sum_{i=1}^{m} a_{ij}^2}, \text{ for } j \in C_j^+, \quad t_j = \sqrt{\frac{1}{m}\sum_{i=1}^{m} \frac{1}{a_{ij}}^2}, \text{ for } j \in C_j^-. \quad (16)$$

Note that the normalization coefficients t_j are constants for fixed j (and similarly k_j), and therefore all three inverse normalizations (13)-(16) are inversions of the same type using the $1/a$ transformation. Note also that when choosing the overall goal of the STB problem, it is necessary to swap the formulas for the benefit and cost criteria. In MCDM problems, as a rule, the general goal is LTB and transformations (13)-(16) are used.

Disadvantages of the $1/a$ transformation:

1) with the $1/a$ transformation, interpretation of the normalized values is difficult. To interpret the normalized values, the connecting term "share of" is used, for example, the share of the largest value. In the case of normalization (13)-(16), the value $1/a_{ij}$ in the numerator of these formulas does not make sense,

2) the first requirement for normalization is to preserve the proportions between the values (preserve dispositions by Eq.(6)). In accordance with Section 2.3, dispositions are invariant with respect to linear data transformations. Inverse normalizations iMax, iSum, iVec are nonlinear, and, therefore, the data and the object are distorted. The acceptability of formulas (13)-(16) is due only to the fact that these distortions are not so great, and in some cases do not affect the final result,

3) the second requirement for normalization is the agreement of the domains of normalized values of different attributes. Agreement of the normalizations for the benefit criteria and the cost criteria is achieved if the domains of the normalized values x_{ij} are comparable. With the $1/a$ transformation, this requirement is met only for the iMax normalization, but for the iSum and iVec methods this is not the case. The domains of values for the benefit and cost criteria are shifted, which leads to a change in the rating.

3.3. Reverse Sorting inversion (ReS)

The Reverse Sorting algorithm is universal, preserves value dispositions, preserves domain positions, and can be applied to both the original (17) and normalized (18) data set for jth criteria [7, 8]:

$$a_{ij} := -a_{ij} + a_j^{max} + a_j^{min}, \text{ for } j \in C_j^-, \quad (17)$$

$$x_{ij} := -x_{ij} + x_j^{max} + x_j^{min}, \text{ for } j \in C_j^-. \quad (18)$$

According to (17) and (18), the following are possible for the normalization option. First, using (17), an inversion is performed for the cost criteria for the original decision matrix, and then one of the linear normalization methods is applied. Another option is to first perform one of the linear normalization methods for the entire decision matrix, and then, using (18), an inversion of the normalized cost criteria values is performed.

In the study [7] it is shown that ReS(Max(a)) = Max(ReS(a)), but ReS(Sum(a)) $\neq$ Sum(ReS(a)) and ReS(Vec(a)) $\neq$ Vec(ReS(a)). The dispositions of the values are preserved, but there is a shift in the domains for the case of Sum(ReS(a)) and Vec(ReS(a)) from the corresponding domains when normalizing Sum(a) and Vec(a). Therefore, it is recommended to use the sequence: normalization-inversion, i.e. formula (18).

The ReS transformation equally inverts both the cost criteria and the benefit criteria. For example, you can invert the values of the cost attributes and then solve the LTB problem, or invert the values of the benefit attributes and then solve the STB problem.

As is known, for optimization problems, the LTB goal is changed to the STB goal by simply changing the sign of the objective function. The known inversions using this simple technique are presented above in Section 3.1:

$$x_{ij} := -x_{ij}, \tag{19}$$

$$x_{ij} := 1 - x_{ij}. \tag{20}$$

In the first case, the cost criteria domain becomes negative in contrast to the positive values for the benefit criteria. In the second case, there is a strong shift in the data — antiphase. It is obvious that ReS is a modification of the above methods. In formulas (17) and (18), an exact shift in the domain of values is performed, which allows preserving the domain of values after the inversion. This is precisely what eliminates the problem of changing the contribution of the attribute to the integral rating under the shift.

According to the results of Section 2.3, the WSM (or TOPSIS) model with the ReS inversion is equivalent to the WSM model with the inversion $x_{ij} := -x_{ij}$ (for $j \in C_j^-$ — cost criteria) and is also equivalent to the WSM with the inversion $x_{ij} := 1 - x_{ij}$ (for $j \in C_j^-$ — cost criteria), where x_{ij} is obtained using any linear normalization method. In all three cases, ΔQ_i are the same, which produces the same ranks of alternatives. In particular, this property excludes the approach of separately accounting for the contribution of benefit and cost criteria in the WSM model (the Ratio System approach) in the form:

$$Q_i = \sum_{j=1}^{g} w_j x_{ij} - \sum_{j=g+1}^{n} w_j x_{ij}, \tag{21}$$

Where g is the number of benefit criteria.

Note also that the ReS transformation is equivalent to the generally accepted inversion for the Max-Min normalization method:

$$x_j^{max} + x_j^{min} - x_{ij} = 1 + 0 - \frac{a_j^{max} - a_{ij}}{a_j^{max} - a_j^{min}} = \frac{a_{ij} - a_j^{min}}{a_j^{max} - a_j^{min}}. \tag{22}$$

A significant advantage of the ReS transformation is the universality of combination with any normalization methods (including normalization of TTB-type criteria and nonlinear normalization) and combination with any methods of aggregation of various attributes.

3.4. Illustrations comparing the nonlinear 1/a inversion and the linear ReS inversion

The final illustration in Fig. 1, 2 and 3 compares the nonlinear inversion $1/a$ and the linear ReS for the Max, Sum and Vec normalizations for the following vector of one of the attributes for 8 alternatives: [1056 2680 1230 1480 1350 2065 1650 1750]. The ReS transformation is performed for the pre-normalized data using formula (17). Figure 1 illustrates the linearity property for ReS and the nonlinearity of the inversion based on the $1/a$ transformation for the Max, Sum and Vec normalization.

The areas of the normalized values are equal only for the Max/ iMax normalization, but there is a slight shift for the Sum/ iSum and Vec/ iVec methods. For the case of iSum, iVec, a shift in the domain of values towards a decrease is observed, which means a decrease in the contribution of this attribute to the overall integral indicator. Figure 2 shows the dispositions between the normalized values for ReS and the nonlinearity of the $1/a$ transformation.

Figure 2 shows the dispositions Δx_i by Eq.(6) between the normalized values for ReS and the nonlinear inversion of iSum. The ReS transformation preserves the dispositions Δx_i of the original data and has a center of symmetry when displayed. For the inversion of iMax, iSum,

iVec, the dispositions do not correspond to the dispositions of the original data; the center of symmetry is absent. For some points, the dispositions differ by up to $\times 2$ times. This is a significant violation of the structure of the original data. Note that Δx_i for iSum and iVec will be the same as for iMax, since the formulas differ only in the denominator, which is constant for fixed j and therefore does not affect Δx_i (see also the values of Δx_i in Figure 2). Note also that ReS inversion together with linear normalization (Sum, Vec, Max-Min,...) preserves the dispositions of the attribute values unchanged according to property (5) or (6). They are the same as for natural values. This is the universality of ReS inversion.

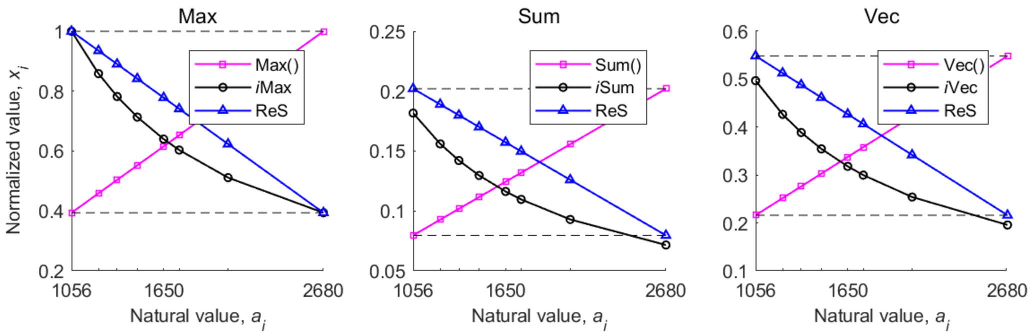

Figure 1: *Normalized and inverted values using the 1/a and ReS technique.*

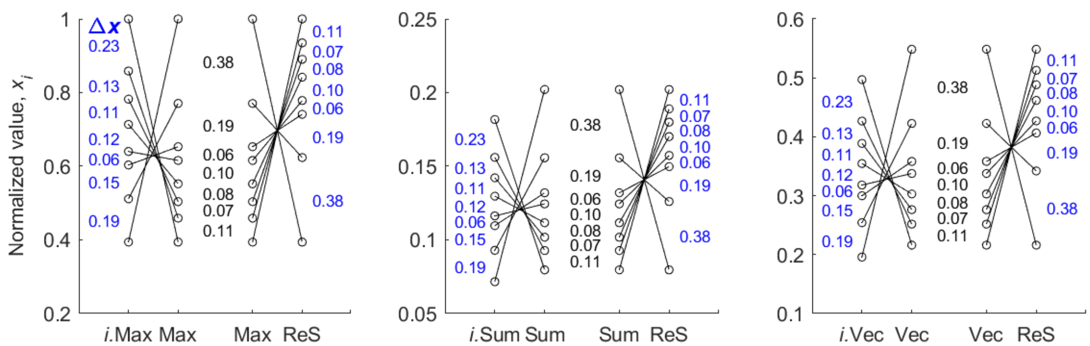

Figure 2: *Distortion of dispositions with a nonlinear inversion of 1/a compared to a linear inversion of ReS.*

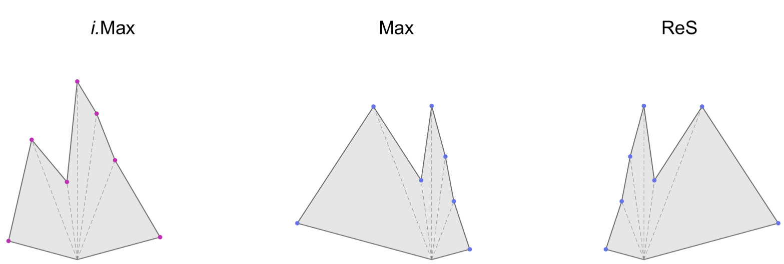

Figure 3: *Polyhedron of attribute dispositions for different normalization/inversion methods.*

Figure 3 shows the polyhedron of attribute dispositions for the original data normalization method Max and the inversion of iMax and ReS. The data polyhedron for Max normalization (which is the same for natural attribute values) coincides with the polyhedron for data obtained with ReS inversion up to mirror symmetry, but differs from the data polyhedron for iMax normalization. Figure 3 demonstrates object scaling during ReS transformation, i.e. the proportions in the data are preserved, unlike the nonlinear transformation based on $1/a$.

3.5. A simple example demonstrating that the widely used $1/a$ transformation for inverting values of attributes introduces structural inconsistencies into normalized data

In addition to the example presented in Section 3.4, we will give a simpler and more illustrative example of incorrectness of the inversion of the $1/a$ type. Let the settlements A, B, and C have 1, 2, and 3 thousand people, respectively. Several more features are defined for these three settlements, which we will omit from consideration. Let the problem be to select one of the settlements for the construction of a logistics warehouse. Let the decision maker give priority to the settlement where the smallest number of people live. Accordingly, "opulation" is the STB criterion. We normalize and invert the data using the Max, iMax, and ReS methods. The results are presented in Table 1.

Settlement	Population (a), 1000 people	x=Max(a)	ReS(a)		iMax		ReS	
			a	$\Delta a,\%$	x	$\Delta x,\%$	x	$\Delta x,\%$
A	1	1/3	3	-	1	-	1	-
B	2	2/3	2	50	1/2	75	2/3	50
C	3	1	1	50	1/3	25	1/3	50

Table 1: *Results of normalization/inversion for the problem of choosing a settlement.*

Now let's compare the ratios Δ of the population in settlement A, B and C obtained by different methods using the formula of disposition (5). The iMax inversion significantly changed the population shares in localities A, B and C after normalization by 75 and 25%, respectively, compared to the original values of 50 and 50%. This simple example demonstrates that the nonlinear inversion of cost attributes of the 1/a type introduces structural contradictions into the normalized data. The current data do not correspond to the attributes of the original object and, therefore, the results of further data processing will be incorrect.

4. Numeric examples

Example 1. The first example shows how significantly the ranking of alternatives A_1 and A_4 for the CODAS/iMax method changed when using the ReS inversion (Table 2 and 3). The original data (Table 2) are from the article promoting the CODAS method [3].

Criteria	C_1^+	C_2^+	C_3^-	C_4^+	C_5^-
A_1	60.00	0.40	2540	500	990
A_2	6.35	0.15	1016	3000	1041
A_3	6.80	0.10	1727	1500	1676
A_4	10.00	0.20	1000	2000	965
A_5	2.50	0.10	560	500	915
A_6	4.50	0.08	1016	350	508
A_7	3.00	0.10	1778	1000	920
$Weight, w_j$	0.036	0.326	0.192	0.326	0.120

Table 2: *Input data: Decision matrix.*

The results of solving the multi-criteria choice problem for various MCDM models are presented in Table 3.

	CODAS/Max/iMax			CODAS/Max/ReS			WSM/Max/iMax			WSM/Max/ReS		
R	Q_i	A_i	ΔQ_i	Q_i	A_i	ΔQ_i	Q_i	A_i	ΔQ_i	Q_i	A_i	ΔQ_i
1	2.19	2	-	2.31	2	-	0.63	2	-	0.68	2	-
2	1.21	1	22.8	1.26	4	26.0	0.56	4	18.1	0.61	4	18.2
3	1.11	4	2.5	0.86	1	9.7	0.53	1	8.7	0.53	1	22.0
4	-0.31	3	32.9	-0.33	3	29.5	0.43	3	25.7	0.47	3	14.9
5	-0.47	5	3.9	-0.78	5	11.1	0.40	5	9.4	0.40	5	20.2
6	-1.63	7	27.0	-1.58	7	19.6	0.32	7	19.9	0.36	7	9.6
7	-2.10	6	10.9	-1.74	6	4.0	0.25	6	18.3	0.30	6	15.1

Table 3: *Rank list (R) of alternatives A_i and performance indicators Q, $\Delta Q(\%)$ for models CODAS/Max/iMax, CODAS/Max/ReS, WSM/Max/iMax, WSM/Max/ReS.*

Even a small difference in the inversion of just one attribute changed the ranks, not to mention the ratings. WSM in this example serves as a test method. The value of ΔQ for the sorted list of ratings shows how much the alternatives of rank p and p+1 differ. With a small difference, such alternatives can be defined as indistinguishable or as alternatives of the same rank [8, 10].

Example 2. The second example shows how significantly the weight of the criteria for the objective MEREC/Max/iMax method has changed in contrast to MEREC/Max/ReS. The original data (Table 4) are from the article promoting the MEREC method [6].

Criteria	C_1^+	C_2^+	C_3^+	C_4^-	C_5^-	C_6^-	C_7^-
A_1	23	264	2.37	0.05	167	8900	8.71
A_2	20	220	2.20	0.04	171	9100	8.23
A_3	17	231	1.98	0.15	192	10800	9.91
A_4	12	210	1.73	0.20	195	12300	10.21
A_5	15	243	2.00	0.14	187	12600	9.34
A_6	14	222	1.89	0.13	180	13200	9.22
A_7	21	262	2.43	0.06	160	10300	8.93
A_8	20	256	2.60	0.07	163	11400	8.44
A_9	19	266	2.10	0.06	157	11200	9.04
A_{10}	8	218	1.94	0.11	190	13400	10.11

Table 4: *Input data: Decision matrix.*

The weights of the criteria are presented in Table 5.

method	w_1	w_2	w_3	w_4	w_5	w_6	w_7
$w^{(1)} : MEREC, Max/iMax$	0.324	0.055	0.086	0.368	0.044	0.077	0.045
$w^{(2)} : MEREC, Max/ReS$	0.266	0.058	0.082	0.409	0.049	0.085	0.050
Relative change,%	-18	5	-5	11	11	10	11

Table 5: *Criteria weights based on the MEREC method, using iMax and ReS inversion.*

For the MEREC method, when using the linear inversion of ReS, the weight of the first and third criteria decreased, and the weight of the remaining criteria increased. The weight changes are significant. Table 6 presents the results of ranking alternatives for the WSM model using different weighting coefficients (according to Table 5) and within the iMax and ReS inversion methods.

R	$w^{(1)}, iMax$			$w^{(1)}, ReS$			$w^{(2)}, iMax$			$w^{(2)}, ReS$		
	Q_i	A_i	ΔQ_i	Q_i	A_i	ΔQ_i	Q_i	A_i	ΔQ_i	Q_i	A_i	ΔQ_i
1	0.930	2	-	0.969	1	-	0.937	2	-	0.967	1	-
2	0.913	1	3.6	0.931	2	7.8	0.905	1	6.7	0.938	2	5.8
3	0.828	7	18.6	0.917	7	2.8	0.818	7	18.3	0.917	7	4.2
4	0.785	9	9.5	0.884	8	6.6	0.780	9	8.1	0.883	8	6.6
5	0.778	8	1.3	0.873	9	2.1	0.766	8	2.8	0.878	9	1.0
6	0.589	3	41.5	0.659	3	43.3	0.571	3	41.2	0.649	3	45.6
7	0.565	5	5.2	0.646	5	2.7	0.553	5	3.9	0.643	5	1.4
8	0.550	6	3.2	0.641	6	1.0	0.541	6	2.4	0.642	6	0.1
9	0.481	10	15.2	0.586	10	11.0	0.488	10	11.2	0.606	10	7.2
10	0.472	4	2.0	0.473	4	22.8	0.463	4	5.3	0.465	4	28.0

Table 6: *Rank list (R) of alternatives A_i and performance indicators Q, ΔQ(%) for models WSM/Max/iMax, WSM/Max/ReS.*

MEREC weighting method use *i*Max and ReS inversion. In this example, the ratings and ranks of alternatives change in the WSM/Max/*i*Max and WSM/Max/ReS models with fixed weights. For the model with different weights: $w^{(1)}$ and $w^{(2)}$ the integral rating of alternatives changes insignificantly, and the ranks of alternatives do not change.

Example 3. The third example shows how significantly the weight of criteria for the objective SECA, Max/*i*Max method has changed in contrast to SECA, Max/ReS method. The original data (Table 4) are from the article promoting the SECA method [2]. The weights of the criteria are presented in Table 7.

method	w_1	w_2	w_3	w_4	w_5	w_6	w_7
$w^{(3)} : SECA, Max/iMax$	0.153	0.151	0.126	0.178	0.118	0.159	0.116
$w^{(4)} : SECA, Max/ReS$	0.156	0.147	0.124	0.173	0.113	0.169	0.118
Relative change,%	2	-3	-2	-3	-4	6	2

Table 7: *Criteria weights based on the SECA method, using iMax and ReS inversion.*

For the SECA method, when using the linear inversion of ReS, the weights of the first, sixth and seventh criteria increased, and the weights of the remaining criteria decreased. The change is small, about 5%, which apparently will not affect the ranking. Table 8 presents the results of ranking within the weight change.

R	$w^{(3)}, iMax$			$w^{(3)}, ReS$			$w^{(4)}, iMax$			$w^{(4)}, ReS$		
	Q_i	A_i	ΔQ_i	Q_i	A_i	ΔQ_i	Q_i	A_i	ΔQ_i	Q_i	A_i	ΔQ_i
1	0.939	1	-	0.967	1	-	0.940	1	-	0.968	1	-
2	0.921	2	5.5	0.932	7	10.3	0.922	2	5.6	0.931	7	10.7
3	0.884	7	11.8	0.924	2	2.4	0.884	7	12.0	0.925	2	2.0
4	0.856	8	8.8	0.912	8	3.4	0.856	8	8.9	0.911	8	4.1
5	0.847	9	2.9	0.895	9	5.0	0.846	9	3.1	0.893	9	5.1
6	0.711	3	42.9	0.750	3	42.4	0.713	3	41.9	0.752	3	41.5
7	0.698	5	3.9	0.741	5	2.5	0.699	5	4.5	0.741	5	3.1
8	0.678	6	6.3	0.725	6	4.8	0.679	6	6.4	0.724	6	5.0
9	0.632	10	14.5	0.684	10	11.9	0.632	10	14.9	0.682	10	12.3
10	0.621	4	3.5	0.625	4	17.2	0.623	4	2.8	0.627	4	16.2

Table 8: *Rank list (R) of alternatives A_i and performance indicators Q, ΔQ(%) for models WSM/Max/iMax, WSM/Max/ReS.*

SECA weighting method use iMax and ReS inversion. In this example, the ratings and ranks of the alternatives change in the WSM/Max/iMax and WSM/Max/ReS models with fixed weights. For the model with different weights: $w^{(3)}$ and $w^{(4)}$ the integral rating of the alternatives changes insignificantly, and the ranks of the alternatives do not change.

5. Conclusions

All the above arguments and examples indicate that the use of nonlinear inversion of the $1/a$ type transforms the data incorrectly. Nonlinear data inversion does not have a reasonable interpretation of values and leads to a violation of mutual distances in the original data. The measurement scales of various attributes are not consistent and there is a shift in the domains of normalized values. The ReS algorithm eliminates all the shortcomings and is recommended for use in inversion of values when coordinating the optimization goals of a multi-criterial problem.

ReS inversion is superior in all respects to existing inversion methods. Cost criteria are inverted consistently with normalized benefit criteria. It should be emphasized that ReS inversion is a logical development of the -x, and (1-x) transformations.

There is no need to exclude from the collection of methods MCDM methods that use the $1/a$ inversion. The authors systematically promote the concept of a multi-model [10], including the triple (F, W, $Norm$) of Normalization/Inversion-Weighting-Aggregation methods. It is sufficient to use ReS inversion instead of the nonlinear $1/a$ inversion. There are no technical problems to replace the normalization/inversion method in any of the MCDM methods. Examples are Max/ReS, Sum/ReS, Vec/ReS and others. Moreover, the ReS transformation is compatible and consistent with any of the nonlinear normalization methods. In this case, changes in the rating of alternatives in many cases, as shown in the examples of our article, will be insignificant and may not lead to changes in the rank of alternatives.

Data inversion is also relevant in objective methods of assessing the weight of criteria based on information contained in the decision matrix. Therefore, in such methods as EWM, MEREC, SECA, CILOS, instead of nonlinear inversion $1/a$, it is recommended to use ReS inversion.

In MCDM methods, it is difficult to check the adequacy of the result, since the optimality criterion is relative. However, our criticism of nonlinear inversion is simple and clear — it is a violation of mutual distances in the data, which distorts the description of the objects of choice.

A more radical result is presented in a recently published [11]. It is shown that as soon as MCDM models use the same linear normalization method and linear inversion, it turns out that a whole range of MCDM models (WSM, RS, MABAC, TOPSIS(L_1), MAIRCA and RAWEC) are equivalent — they always have the same ranking of alternatives and the same dispositions of the performance indicator.

One of the unresolved issues remains the question of which data set should the ReS inversion be applied to, the original data, according to (17), or the normalized data, according to (18). Although in both cases the dispositions in the data are preserved, the values are shifted. The authors also believe that further research is required in the direction of synthesizing a solution based on the ratings obtained in different MCDM models combining different combinations of the Normalization/Inversion-Weighting- and Aggregation methods. The tendency to synthesize solutions based on a large number of MCDM models is based on the assumption that each method includes different relationships between alternatives, which increases the overall information content and improves the reliability of the result. This premise suggests defining a list of acceptable methods for solving a specific problem. These methods should be independent and reflect different relationships between alternatives.

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