A new search direction for full-Newton step infeasible interior-point method in linear optimization

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Abstract. In this work, we investigate a full Newton step infeasible interior-point method for linear optimization based on a new search direction which is obtained from an algebraic equivalent transformation of the central path system. Furthemore, we prove that the proposed method obtains an ε-optimal solution to the original problem in polynomial time.

Keywords: Infeasible interior-point methods, linear optimization, new search directions, polynomial complexity

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1. Introduction

One of the most powerful methods for solving linear optimization (LO) problems is interior-point method (IPM). Since the pioneering algorithm of Karmarkar [6] for LO, a lot of research has been done on IPMs. At the same time, primal-dual IPMs have attracted more attention. These methods use Newton's directions, which are closely related to the primal-dual logarithmic barrier function. In the design of primal-dual IPMs, search directions play an essential role. Peng et al. [20, 21] replaced the logarithmic barrier function with a so-called self-regular barrier function and modified the search direction accordingly. Then, they derived a large-update method for which the theoretical iteration bound is $O(\sqrt{n}\log n \log \frac{n}{\varepsilon})$. Bai et al. [2] proposed a new large-update method based on a so-called kernel function (KF), which is not self-regular and has an $O(\sqrt{n}\log n \log \frac{n}{\varepsilon})$ iteration bound.

Primal-dual IPMs are divided into two cathegories: feasible IPMs and infeasible IPMs (IIPMs). Feasible IPMs start from a strictly feasible point, which is difficult to find. In that case an IIPM should be used. These methods have the advantage that they can start with an arbitrary positive point and simultaneously strive to reach feasibility and optimality, simultaneously. IIPMs were first introduced by Lustig [19] and Tanabe [25].

The first full-Newton step feasible IPM for LO was presented by Roos et al. [24]. Darvay [3] proposed a new technique to find a set of search directions. Using an Algebraic Equivalent Transformation (AET) of the nonlinear equations of the central path system, the author designed a full-Newton step fesible IPM for LO with iteration bound $O(\sqrt{n}\log \frac{n}{\varepsilon})$. The method is extended to convex quadratic optimization (CQO) [1], second-order cone optimization (SOCO) [26], symmetric optimization (SO) [27] and the Cartesian $P_*(\kappa)$ linear complementarity problem (LCP) [28]. Considering the new search direction created by the $\psi(t) = \sqrt{t}/2(1+\sqrt{t})$ function in the AET technique, Kheirfam and Haghighi [17] proposed a full-Newton step feasible IPM for $P_*(\kappa)$-LCPs. According to the $\psi(t) = t - \sqrt{t}$ function used in the AET technique, Darvay

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et al. [4] introduced a full-Newton step feasible IPM for LO. Darvay and Takács [5] considered the $\psi(t) = t^2$ function in the new AET $\psi(\frac{xs}{\mu}) = \psi(\sqrt{\frac{xs}{\mu}})$ and proposed a full-Newton step feasible IPM for LO.

Roos [22] proposed a full-Newton step IIPM for LO, which is a generalization of the IPM analyzed in [24]. Some generalizations and versions of this method are presented in [8, 9, 10, 15, 18, 30]. Roos [23] introduced an improved version of the method for LO that does not require centering steps, while the aforementioned methods require several (at most three) centering steps in each (main) iteration. Kheirfam extended this method to HLCP [11], the Cartesian $P_*(\kappa)$-LCP [12], the convex quadratic symmetric cone optimization (CQSCO) [13] and SO [14]. Kheirfam [7] proposed an infeasible version of the method presented in [4] for SDLCP.

Motivated by the aforementioned works, in this paper we consider a full-Newton step IIPM for LO based on the AET strategy introduced in [5]. By applying Newton's method in the transformed system, the search directions are obtained. We prove the convergence of the proposed algorithm and derive its iteration bound.

The paper is organized as follows. In the next section, we recall the problem pair (P) and (D). We state that the perturbed problems related to (P) and (D) and then present the central path. In Section 3, we introduce the search directions of the IIPM for LO and then present the algorithm. Section 4 is dedicated to the complexity analysis of the proposed method. In Section 5, some concluding remarks are followed.

2. Preliminaries

Consider the LO problem in the following standard form

$$(P) \qquad \min\ \{c^T x:\ Ax = b,\ x \geq 0\},$$

where $A \in R^{m\times n}$ with $rank(A) = m, b \in R^m$ and $c \in R^n$. Its dual problem is in the following standard form:

$$(D) \qquad \max\ \{b^T y:\ A^T y + s = c,\ s \geq 0\},$$

where $y \in R^m$ and $s \in R^n$. In accordance with the routine of IIPMs, we consider the starting point $(x^0, y^0, s^0) = \xi(e, 0, e)$ such that $\|(x^*; s^*)\|_\infty \leq \xi$ for some primal-dual optimal solution (x^*, y^*, s^*), where $e = (1, ..., 1)^T$ and $0 < \xi \in R$. It should be noted that for the optimal solution (x^*, y^*, s^*) the inequality $\|(x^*; s^*)\|_\infty \leq \xi$ is true if and only if

$$0 \leq x^* \leq \xi e, \quad 0 \leq s^* \leq \xi e. \tag{1}$$

For an IIPM, a triple (x, y, s) is said to be an ε-solution of (P) and (D) if

$$\max\left\{x^T s, \|b - Ax\|, \|c - A^T y - s\|\right\} \leq \varepsilon,$$

where ε is a accuracy parameter. Following [22], for any $0 < \nu \leq 1$ we consider the perturbed problem pair (P_ν) and (D_ν) as follows:

$$(P_\nu) \qquad \min\ \{(c - \nu r_c^0)^T x:\ b - Ax = \nu r_b^0,\ x \geq 0\},$$

$$(D_\nu) \qquad \max\ \{(b - \nu r_b^0)^T y:\ c - A^T y - s = \nu r_c^0,\ s \geq 0\},$$

where $r_b^0 := b - A\xi e$ and $r_c^0 := c - \xi e$. It is easy to see that $(x^0, y^0, s^0) = \xi(e, 0, e)$ is a feasible solution of the problem pair (P_ν) and (D_ν) if $\nu = 1$. We conclude that the problem pair (P_ν) and (D_ν) satisfies the interior point condition (IPC) if $\nu = 1$. We recall the following lemma.

Lemma 1. (Theorem 5.13 in [29]) *The original problems,* (P) *and* (D) *are feasible if and only if for each* ν *satisfying* $0 < \nu \leq 1$ *the perturbed problems* (P_ν) *and* (D_ν) *satisfy the IPC.*

In the view of Lemma 1, we assume that the original problem pair (P) and (D) is feasible and $\nu \in (0, 1]$, the central path of the perturbed pair (P_ν) and (D_ν) exists; that is,

$$\begin{aligned} b - Ax &= \nu r_b^0, \quad x \geq 0, \\ c - A^T y - s &= \nu r_c^0, \quad s \geq 0, \\ xs &= \mu e, \end{aligned} \tag{2}$$

has a unique solution $(x(\mu, \nu), y(\mu, \nu), s(\mu, \nu))$, for every $\mu > 0$. This solution consists of the μ-centers of the perturbed problems (P_ν) and (D_ν). Note that for $x, s > 0$ and $\mu > 0$ from the third equation of system (2) we deduce that

$$xs = \mu e \Leftrightarrow \frac{xs}{\mu} = e \Leftrightarrow \sqrt{\frac{xs}{\mu}} = e \Leftrightarrow \frac{xs}{\mu} = \sqrt{\frac{xs}{\mu}}. \tag{3}$$

Now the perturbed central path can be equivalently expressed as follows

$$\begin{aligned} b - Ax &= \nu r_b^0, \quad x \geq 0, \\ c - A^T y - s &= \nu r_c^0, \quad s \geq 0, \\ \frac{xs}{\mu} &= \sqrt{\frac{xs}{\mu}}. \end{aligned} \tag{4}$$

In the following, the parameters μ and ν always satisfy the relation $\mu = \nu\mu^0 = \nu\xi^2$.

3. Search directions

As mentioned previously (see Section 1), large-update IPMs based on KFs are presented to solve LO.

A twice continuously differentiable function $\psi : R_{++} \to R_+$ is a KF if:

$$i)\ \psi(1) = \psi^{'}(1) = 0, \quad ii)\ \psi^{''}(t) > 0 \text{ for all } t > 0.$$

The KF is called coercive if $\lim_{t \downarrow 0} \psi(t) = \lim_{t \to +\infty} \psi(t) = +\infty$.

The main idea of full-Newton step IIPMs based on KFs is to determine the search directions $(\Delta x, \Delta y, \Delta s)$ such that

$$\begin{aligned} A\Delta x &= \theta\nu r_b^0, \\ A^T \Delta y + \Delta s &= \theta\nu r_c^0, \\ s\Delta x + x\Delta s &= -\sqrt{\mu xs}\ \nabla\Psi\Big(\sqrt{\tfrac{xs}{\mu}}\Big), \end{aligned}$$

where the generalized barrier function $\Psi(t), t \in R^n_{++}$ is in the following form:

$$\Psi(t) = \sum_{i=1}^{n} \psi(t_i).$$

In the what following, we propose an AET-based method for introducing search directions.

In accordance with Darvay's idea, we consider the function ψ defined and continuously differentiable on the interval (k^2, ∞), where $0 \leq k < 1$, such that $2t\psi^{'}(t^2) - \psi^{'}(t) > 0, \forall t > k^2$. Now, if we apply the AET strategy to (4), we get

$$b - Ax = \nu r_b^0, \quad x \geq 0, \tag{5}$$

$$c - A^T y - s = \nu r_c^0, \quad s \geq 0, \tag{6}$$

$$\psi\Big(\frac{xs}{\mu}\Big) = \psi\Big(\sqrt{\frac{xs}{\mu}}\Big). \tag{7}$$

Let (x, y, s) be a feasible solution of the perturbed pair (P_ν) and (D_ν). We consider the notation

$$f(x,y,s) = \begin{bmatrix} \nu^+ r_b^0 - b + Ax \\ \nu^+ r_c^0 - c + A^T y + s \\ \psi\Big(\dfrac{xs}{\mu}\Big) - \psi\Big(\sqrt{\dfrac{xs}{\mu}}\Big) \end{bmatrix} = 0,$$

where $\nu^+ = (1-\theta)\nu$ and $\theta \in (0,1)$. By applying Newton's method in this system, we have

$$J_f(x,y,s) \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = -f(x,y,s),$$

where $J_f(x,y,s)$ denotes the Jacobian matrix of f at (x,y,s). After some computations, we obtain the following system:

$$\begin{aligned} A\Delta x &= \theta\nu r_b^0, \\ A^T\Delta y + \Delta s &= \theta\nu r_c^0, \\ \tfrac{1}{\mu}\big(s\Delta x + x\Delta s\big) &= \frac{-\psi(\frac{xs}{\mu}) + \psi(\sqrt{\frac{xs}{\mu}})}{\psi'(\frac{xs}{\mu}) - \frac{1}{2\sqrt{\frac{xs}{\mu}}}\psi'(\sqrt{\frac{xs}{\mu}})}. \end{aligned} \tag{8}$$

We define the following scaled search directions

$$d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s}, \quad \text{where } v = \sqrt{\frac{xs}{\mu}}. \tag{9}$$

Then we rewrite the system (8) as follows:

$$\begin{aligned} \bar{A} d_x &= \theta\nu r_b^0, \\ \bar{A}^T \tfrac{\Delta y}{\mu} + d_s &= \theta\nu v s^{-1} r_c^0, \\ d_x + d_s &= p_v, \end{aligned} \tag{10}$$

where

$$p_v := \frac{2\psi(v) - 2\psi(v^2)}{2v\psi'(v^2) - \psi'(v)}, \quad \text{and} \quad \bar{A} := A\mathrm{diag}\Big(\frac{x}{v}\Big).$$

If we use the function $\psi : (\frac{1}{\sqrt{2}}, \infty) \to \mathbb{R}, \psi(t) = t^2$ introduced in [5], then we obtain

$$p_v = \frac{v - v^3}{2v^2 - e}. \tag{11}$$

After a full-Newton step, the new iterate is given by

$$x_+ := x + \Delta x, \quad y_+ := y + \Delta y, \quad s_+ := s + \Delta s. \tag{12}$$

In addition, in each iteration of the algorithm, a quantity is needed to measure the distance of an iterate from the central path. For this purpose, We consider the proximity measure defined by the following quantity

$$\delta(v) := \delta(x, s; \mu) = \frac{\|p_v\|}{2} = \frac{1}{2}\Big\|\frac{v - v^3}{2v^2 - e}\Big\|, \tag{13}$$

which was first proposed for a feasible IPM in [5].

Let $q_v = d_x - d_s$. Then

$$d_x = \frac{p_v + q_v}{2}, \quad d_s = \frac{p_v - q_v}{2}, \quad d_x d_s = \frac{p_v^2 - q_v^2}{4}, \tag{14}$$

and

$$\frac{\|q_v\|^2}{4} = \frac{\|d_x - d_s\|^2}{4} = \frac{\|d_x + d_s\|^2}{4} - d_x^T d_s = \frac{\|p_v\|^2}{4} - d_x^T d_s. \tag{15}$$

Suppose that for some $\mu \in (0, \mu^0]$, our algorithm starts from a feasible solution (x, y, s) of the problem pair (P_ν) and (D_ν) with $\nu = \frac{\mu}{\mu^0}$, and such that $\delta(x, s; \mu) \leq \tau, \tau \in (0, 1)$. Then, the algorithm finds a feasible solution (x_+, y_+, s_+) of (P_{ν^+}) and (D_{ν^+}), where $\nu^+ = (1-\theta)\nu, \theta \in (0, 1)$. In this case, μ is reduced to $\mu^+ = (1-\theta)\mu$ and such that $\delta(x_+, s_+; \mu^+) \leq \tau$. This procedure is repeated until an ε-solution is found. We are now in a position to express the theoretical framework of the infeasible interior-point algorithm as follows:

Algorithm1 : an infeasible interior − point algorithm

Input :
  Accuracy parameter $\epsilon > 0$;
  barrier update parameter θ, $0 < \theta < 1$;
  threshold parameter $\tau > 0$.
begin
  $x := \xi e;\ y := 0;\ s := \xi e;\ \mu := \nu \xi^2;\ \nu = 1;$
 while $\max(x^T s, \|r_b\|, \|r_c\|) > \epsilon$ **do**
  begin
   solve the system (10) and use (9) to obtain $(\Delta x, \Delta y, \Delta s)$;
    $(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$;
   update of μ and ν :
     $\mu := (1-\theta)\mu$;
     $\nu := (1-\theta)\nu$;
  end
end.

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4. Analysis of the algorithm

Here, we will prove that Algorithm 1 is well defined. The main goal of our analysis is to find some values for the parameters τ and θ such that $x_+ > 0$ and $s_+ > 0$, and we have $\delta(x_+, s_+; \mu^+) \leq \tau$. In the following section, after an iteration of the algorithm we obtain an upper bound for the proximity measure.

4.1. Upper bound for $\delta(v^+)$

In the next lemma, we give a condition on the proximity measure that guarantees the feasibility of a full-Newton step. In what follows, we use the notation $\omega = \frac{1}{2}\left(\|d_x\|^2 + \|d_s\|^2\right)$.

Lemma 2. *The iterate (x_+, y_+, s_+) with $v > \frac{1}{\sqrt{2}}e$ is strictly feasible if $2\delta(v)^2 + \omega < 1$.*

Proof. Let $0 \leq \alpha \leq 1$. We define $x(\alpha) := x + \alpha\Delta x$ and $s(\alpha) := s + \alpha\Delta s$. Using (9), the third equation of (10) and (14) one can finds

$$\begin{aligned}\frac{x(\alpha)s(\alpha)}{\mu} &= \frac{xs}{v^2}(v+\alpha d_x)(v+\alpha d_s) = v^2+\alpha v(d_x+d_s)+\alpha^2 d_x d_s \\ &= (1-\alpha)v^2+\alpha(v^2+vp_v)+\alpha^2\Big(\frac{p_v^2-q_v^2}{4}\Big) \\ &\geq (1-\alpha)v^2+\alpha^2 e+\alpha^2\frac{p_v^2}{4}-\alpha^2\frac{q_v^2}{4},\end{aligned} \tag{16}$$

where the inequality is due to $\alpha \geq \alpha^2$ and the following inequality:

$$v^2+vp_v-e = v^2+\frac{v^2-v^4}{2v^2-e}-e = \frac{v^4}{2v^2-e}-e = \frac{(v^2-e)^2}{2v^2-e} \geq 0. \tag{17}$$

The inequality $x(\alpha)s(\alpha) > 0$ holds if

$$\begin{aligned}\Big\| -\frac{p_v^2}{4}+\frac{q_v^2}{4}\Big\|_\infty &\leq \Big\|\frac{p_v^2}{4}\Big\|_\infty+\Big\|\frac{q_v^2}{4}\Big\|_\infty \leq \frac{\|p_v\|^2}{4}+\frac{\|q_v\|^2}{4} \\ &= 2\delta(v)^2-d_x^T d_s \leq 2\delta(v)^2+\|d_x\|\|d_s\| \leq 2\delta(v)^2+\omega < 1,\end{aligned}$$

where the equality is due to (15), the third inequality uses from the Cauchy-Schwarz inequality and the last inequality holds due to the assumption of the lemma. Thus $x(\alpha)s(\alpha) > 0$, for $0 \leq \alpha \leq 1$; $x(\alpha)$ and $s(\alpha)$ do not change sign on the interval $[0,1]$. Consequently, $x(0) = x > 0$ and $s(0) = s > 0$ yields $x(1) = x_+ > 0$ and $s(1) = s_+ > 0$. Thus the proof is completed. □

In correspondence to the definition (13), we have

$$\delta(v_+) = \delta(x_+, s_+; \mu^+) = \frac{1}{2}\Big\|\frac{v_+-v_+^3}{2v_+^2-e}\Big\|, \text{ where } v_+ = \sqrt{\frac{x_+s_+}{\mu^+}}.$$

Lemma 3. *Let $\delta(v)^2+\omega < \frac{1}{2}(1+\theta)$ and $v > \frac{1}{\sqrt{2}}e$. Then, $v_+ > \frac{1}{\sqrt{2}}e$ and*

$$\delta(v_+) \leq \frac{\sqrt{1-\delta(v)^2-\omega}\big(\theta\sqrt{n}+10\delta(v)^2+\omega\big)}{2\sqrt{1-\theta}(2(1-\delta(v)^2-\omega)-(1-\theta))}.$$

Proof. Let $\alpha = 1$. Then from (16) it follows that

$$\begin{aligned}v_+^2 = \frac{x_+s_+}{\mu^+} &= \frac{v^2+vp_v+\frac{p_v^2}{4}-\frac{q_v^2}{4}}{1-\theta} = \frac{e+\frac{(v^2-e)^2}{2v^2-e}+\frac{p_v^2}{4}-\frac{q_v^2}{4}}{1-\theta} \\ &= \frac{e+\big(\frac{9v^2-4e}{v^2}\big)\frac{p_v^2}{4}-\frac{q_v^2}{4}}{1-\theta} \geq \frac{e-\frac{q_v^2}{4}}{1-\theta},\end{aligned}$$

where the second equality is due to (17) and the inequality follows from the fact that $9v^2-4e \geq 0.5e > 0$. Consequently, we have

$$\min(v_+) \geq \sqrt{\frac{1-\frac{1}{4}\|q_v\|_\infty^2}{1-\theta}} \geq \sqrt{\frac{1-\frac{1}{4}\|q_v\|^2}{1-\theta}} \geq \sqrt{\frac{1-\delta(v)^2-\omega}{1-\theta}}, \tag{18}$$

where the last inequality follows from (15) and the Cauchy-Schwarz inequality.

From $\delta(v)^2 + \omega < \frac{1}{2}(1+\theta)$ it follows that $\min(v_+) > \frac{1}{\sqrt{2}}$, hence $v_+ > \frac{1}{\sqrt{2}}e$. Now, we have

$$\begin{aligned}\delta(v_+) &= \frac{1}{2}\left\|\frac{v_+ - v_+^3}{2v_+^2 - e}\right\| = \frac{1}{2}\left\|\frac{v_+}{2v_+^2 - e}(e - v_+^2)\right\| \\ &\leq \frac{\min(v_+)}{2(2\min(v_+)^2 - 1)}\left\|e - v_+^2\right\| \\ &\leq \frac{\sqrt{(1-\theta)(1-\delta(v)^2 - \omega)}}{2(2(1-\delta(v)^2-\omega)-(1-\theta))}\left\|e - v_+^2\right\|. \end{aligned} \tag{19}$$

On the other hand, one has

$$\begin{aligned}\left\|e - v_+^2\right\| &= \left\|\frac{e + \left(\frac{9v^2-4e}{v^2}\right)\frac{p_v^2}{4} - \frac{q_v^2}{4}}{1-\theta} - e\right\| \\ &\leq \frac{1}{1-\theta}\left(\|\theta e\| + \left\|\left(\frac{9v^2-4e}{v^2}\right)\frac{p_v^2}{4} - \frac{q_v^2}{4}\right\|\right) \\ &\leq \frac{1}{1-\theta}\left(\theta\sqrt{n} + 9\frac{\|p_v\|^2}{4} + \frac{\|q_v\|^2}{4}\right) \\ &= \frac{1}{1-\theta}\left(\theta\sqrt{n} + 10\delta(v)^2 + \omega\right).\end{aligned}$$

Substituting this bound into (19) gives us exactly the desired result. Thus the proof is completed. □

4.2. Upper bound for ω

Following [23], let $\mathcal{N} := \{\zeta : \bar{A}\zeta = 0\}$ denote the null space of the matrix $\bar{A}$. Then, the affine space $\{\zeta : \bar{A}\zeta = \theta\nu r_b^0\}$ is equal to $\mathcal{N} + d_x$. Since the row space of $\bar{A}$ is the orthogonal complement $\mathcal{N}^\perp$ of $\mathcal{N}$, thus $d_s \in \theta\nu v s^{-1} r_c^0 + \mathcal{N}^\perp$. Also note that $\mathcal{N} \cap \mathcal{N}^\perp = \{0\}$, and as a consequence the affine spaces $\mathcal{N} + d_x$ and $\mathcal{N}^\perp + d_s$ meet in a unique point q. Applying a similar argument to Lemma 3.4 in [23], we can conclude

$$2\omega \leq \|q\|^2 + \left(\|q\| + \left\|\frac{v - v^3}{2v^2 - e}\right\|\right)^2 = \|q\|^2 + \left(\|q\| + 2\delta(v)\right)^2. \tag{20}$$

Again from [23], we have

$$\|q\| \leq \frac{\theta\left(n + \|v\|^2\right)}{\min(v)}. \tag{21}$$

By definition (13) , we have

$$2\delta(v) = \left\|\frac{v - v^3}{2v^2 - e}\right\| = \left\|\frac{v^2 + v}{2v^2 - e}(e - v)\right\| \geq \frac{1}{2}\left\|e - v\right\| \geq \frac{1}{2}\left(\|v\| - \|e\|\right),$$

which implies

$$\|v\| \leq \|e\| + 4\delta(v) = \sqrt{n} + 4\delta(v).$$

Furthermore, we have

$$4\delta(v) \geq \|e - v\| \geq |1 - v_i|, i = 1, \ldots, n.$$

This gives $\min(v) \geq 1 - 4\delta(v)$. Combining these two inequalities with (21), we will get

$$\|q\| \leq \frac{\theta\left(n + \left(\sqrt{n} + 4\delta(v)\right)^2\right)}{1 - 4\delta(v)}. \tag{22}$$

4.3. Values for θ and τ

In this section, we require to find values θ and τ such that if $\delta(v) \leq \tau$ holds, then $\delta(v_+) \leq \tau$. From Lemma 3, it suffices to have

$$\frac{\sqrt{1-\delta(v)^2-\omega}\big(\theta\sqrt{n}+10\delta(v)^2+\omega\big)}{2\sqrt{1-\theta}(2(1-\delta(v)^2-\omega)-(1-\theta))} \leq \tau, \tag{23}$$

provided that $\delta(v)^2+\omega < \frac{1}{2}(1+\theta)$. One can easily see the right-hand-side of (22) is monotonically increasing with respect to $\delta(v) < 1$. Therefore, invoking $\delta(v) \leq \tau$, we have

$$\|q\| \leq \frac{\theta\Big(n+\big(\sqrt{n}+4\tau\big)^2\Big)}{1-4\tau}.$$

By substituting the above result into (20) and using again $\delta(v) \leq \tau$, we obtain

$$\omega \leq \frac{1}{2}\left[\left(\frac{\theta\big(n+\big(\sqrt{n}+4\tau\big)^2\big)}{1-4\tau}\right)^2+\left(\frac{\theta\big(n+\big(\sqrt{n}+4\tau\big)^2\big)}{1-4\tau}+2\tau\right)^2\right] =: f(\tau)$$

We claim that

$$\chi(t) := \frac{\sqrt{1-t}}{2(1-t)-(1-\theta)}, \quad 0 \leq t \leq \frac{1}{2}(1+\theta), \tag{24}$$

is increasing. Hence, $0 \leq \delta(v)^2+\omega \leq \tau^2+f(\tau)$ implies $\chi(\delta(v)^2+\omega) \leq \chi(\tau^2+f(\tau))$. Therefore, $\delta(v)^2+\omega \leq \frac{1}{2}(1+\theta)$ and (23) will certainly hold if

$$\tau^2+f(\tau) \leq \frac{1}{2}(1+\theta), \quad y(\tau) := \frac{\chi(\tau^2+f(\tau))(\theta\sqrt{n}+10\tau^2+f(\tau))}{2\sqrt{1-\theta}} \leq \tau.$$

If we take $\tau = \frac{1}{16}$ and $\theta = \frac{1}{20n}, n \geq 4$, then $\tau^2+f(\tau) \leq 0.0534 < 0.5000 \leq \frac{1}{2}(1+\theta)$ and $y(\tau) \leq 0.0614 < 0.0625 = \frac{1}{16}$. Hence, we may state the following result.

Lemma 4. *If $\tau = \frac{1}{16}$ and $\theta = \frac{1}{20n}, n \geq 4$, then $\delta(v) \leq \tau$ implies $\delta(v_+) \leq \tau$.*

4.4. Complexity analysis

Lemma 4 shows that Algorithm 1 is well-defined, in the sense that the property $\delta(x, s; \mu) := \delta(v) \leq \tau$ is preserved in all iterations.

In each main iteration, both the barrier parameter μ and the norms of the residual vectors are reduced by a factor of $1-\theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta}\log\frac{\max\{n\xi^2, \|r_b^0\|, \|r_c^0\|\}}{\varepsilon}.$$

Now, we state our main result.

Theorem 1. *If (P) and (D) are feasible and $\xi > 0$ such that $\|(x^*; s^*)\|_\infty \leq \xi$ for some optimal solutions x^* of (P) and (y^*, s^*) of (D), then after at most*

$$20n\log\frac{\max\{n\xi^2, \|r_b^0\|, \|r_c^0\|\}}{\epsilon}$$

iterations, the algorithm finds an ϵ-optimal solution of (P) and (D).

5. Concluding remarks

The method presented in this paper is a full-Newton step IIPM for LO based on the AET introduced in [5]. The method uses only one feasibility step in each iteration. The analysis of our method differs from existing IIPMs based on AET because it uses a different AET. The obtained complexity bound corresponds to the currently best-known theoretical iteration bound for IIPMs. In Table 1 we compare the obtained complexity results with the complexity bounds for IIPMs in [23, 16, 11, 12, 13, 14].

	(τ, θ)	iterations
Algorithm 1	$(\frac{1}{16}, \frac{1}{20n})(n \geq 4)$	$20n \log \frac{\max\{n\xi^2, \|r_b^0\|, \|r_c^0\|\}}{\epsilon}$
Algorithm in [23]	$(\frac{1}{5}, \frac{1}{8n})(n \geq 2)$	$8n \log \frac{\max\{n\xi^2, \|r_b^0\|, \|r_c^0\|\}}{\epsilon}$
Algorithm in [16]	$(\frac{1}{16}, \frac{1}{22n})(n \geq 2)$	$22n \log \frac{\max\{n\xi^2, \|r_b^0\|, \|r_c^0\|\}}{\epsilon}$
Algorithm in [11]	$(\frac{1}{6(1+2\kappa)}, \frac{1}{27n(1+2\kappa)^2})$	$27n(1+2\kappa)^2 \log \frac{\max\{(x^0)^T s^0, \|r_q\|\|\}}{\epsilon}$
Algorithm in [12]	$(\frac{1}{8(1+2\kappa)}, \frac{1}{44r(1+2\kappa)(1+4\kappa)})$	$44r(1+2\kappa)(1+4\kappa) \log \frac{\max\{tr(x^0 \circ s^0), \|r_q^0\|_F\|\}}{\epsilon}$
Algorithm in [13]	$(\frac{1}{7}, \frac{1}{11r})(r \geq 2)$	$11r \log \frac{\max\{r\xi^2, \|r_b^0\|_F, \|r_c^0\|_F\}}{\epsilon}$
Algorithm in [14]	$(\frac{1}{16}, \frac{1}{53r})$	$53r \log \frac{\max\{tr(x^0 \circ s^0), \|r_p^0\|_F\|, \|r_d^0\|_F\}}{\epsilon}$

Table 1. *Comparison of obtained complexity results.*

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