Min-max optimal public service system design

Marek Kvet[1,*] **and Jaroslav Janáček**[2]

[1] *University of Žilina, University Science Park,*
Univerzitná 8215/1, 010 26 Žilina, Slovakia
E-mail: ⟨*marek.kvet@uniza.sk*⟩

[2] *University of Žilina, Faculty of Management Science and Informatics,*
Univerzitná 8215/1, 010 26 Žilina, Slovakia
E-mail: ⟨*jaroslav.janacek@fri.uniza.sk*⟩

Abstract. This paper deals with designing a fair public service system. To achieve fairness, various schemes are be applied. The strongest criterion in the process is minimization of disutility of the worst situated users and then optimization of disutility of the better situated users under the condition that disutility of the worst situated users does not worsen, otherwise called lexicographical minimization. Focusing on the first step, this paper endeavours to find an effective solution to the weighted p-median problem based on radial formulation. Attempts at solving real instances when using a location-allocation model often fail due to enormous computational time or huge memory demands. Radial formulation can be implemented using commercial optimisation software. The main goal of this study is to show that the suitability solving of the min-max optimal public service system design can save computational time.

Key words: min-max approach, radial formulation, public service system

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1. Introduction

A classical approach to optimal public service system design usually locates a limited number of service centre positions from a given finite set of possible locations in order to minimise the sum of distances from a particular system user to the nearest located service centre. Regardless of the type of distance, representing travelling time, cost or other form of social costs, this approach usually involves the weighted p-median problem [8, 9]. Complexity of the p-median problem and the necessity to solve large instances of the problem has led to the search for a suitable algorithm suitable to the task. The study found that the radial formulation approach to the problem considerably facilitates

* Corresponding author.

establishing the associated solution process [1, 5, 6]. Together with this research, attention was given to so-called approximate approaches, which make use of commercial IP-solvers and radial formulation with a homogenous system of radii related to individual users [7, 12]. These approaches are called approximate not because of the solving tool, but the slight impreciseness connected with rounding off the distance values to the values from the set of so-called dividing points. The approximate approach used for the optimal public service system design proved to be a suitable and adequately precise tool when faced with the system designer's lack of time necessary for developing a proprietary informatics-based decision support tool. As the public service system designer must often face objections from potential system users concerning unfair service accessibility provided by the designed system, approaches to fair optimal public service system design were broadly studied. The issue of fairness in general emerges whenever limited resources should be fairly distributed among participants [2, 13 and 14]. A plethora of fairness schemes was studied, but the best applicable scheme in public service system design is the so-called lexicographic min-max criterion. In applying this criterion, the distance or generally disutility of the worst situated user is minimised first and only then that of the second worst situated user, unless the previously achieved disutility of the worst situated users worsens. This step-by-step approach is applied to the remaining users. Various approaches to lexicographic minimisation were developed [3, 15]. The effective utilisation of the above approaches is based on partitioning the range of all possible disutility values, as perceived by a user. The similarity of the range of the partitioning and dividing points in the radial approach led us to the idea of employing homogenous radial formulation for solving the lexicographic location problem.

This paper focuses on the first step of the lexicographic approach, which consists in solving the min-max optimal public service system design, where only the worst situated user's disutility is minimised.

The remainder of this paper includes the following sections: Section 2 contains a description of three different effective approaches for searching for a min-max optimal design of a public service system. Section 3 reports on performed experiments and the results. Finally, Section 4 summarizes the benefits of this study and provides some concluding remarks.

2. Min-max optimal design of public service system

2.1. Location-allocation approach to min-max optimization

The problem involving the min-max public service system design can be described by the following denotations. Symbol J denotes the set of user locations and symbol I the set of possible service centre locations. Constant b_j is the number of users, who share location j. To solve the problem, a maximum of p locations is chosen from I so that the maximal disutility perceived by the worst situated user is minimal. The value of a user's disutility is given by the mutual positions of a user's location and the location of the centre providing the service. The assumption is that the user's disutility grows with distance increasing between the user and the service centre. Disutility describing the distance between locations i and j is denoted as d_{ij}. Decisions that determine the designed public service system are further modelled by introducing decision variables. The variable $y_i \in \{0, 1\}$ models the decision at the service centre located at $i \in I$. The variable has the value of 1 if a service centre is located at i or 0 if otherwise. Furthermore, allocation variables $z_{ij} \in \{0, 1\}$ are introduced for each $i \in I$ and $j \in J$ to assign a user location j to the service centre location i ($z_{ij} = 1$), which provides the service to the user. Thus, the location-allocation model can be formulated as follows.

$$Minimize \quad h \tag{1}$$

$$Subject\ to: \quad \sum_{i \in I} z_{ij} = 1 \quad for\ j \in J \tag{2}$$

$$z_{ij} \le y_i \quad for\ i \in I, j \in J \tag{3}$$

$$\sum_{i \in I} y_i \le p \tag{4}$$

$$\sum_{i \in I} d_{ij} z_{ij} \le h \quad for\ j \in J \tag{5}$$

$$y_i \in \{0,1\} \quad for\ i \in I \tag{6}$$

$$z_{ij} \in \{0,1\} \quad for\ i \in I, j \in J \tag{7}$$

$$h \ge 0 \tag{8}$$

In this model, the objective function (1) represented by the single variable h gives the upper bound of all perceived disutility values. The constraints (2) ensure that each user location is assigned to exactly one of the possible service centres. The link-up constraints (3) assure that the locations of users are assigned only to the located service centres and the constraint (4) limits the number of located service centres by p. The link-up constraints (5) ensure that each perceived disutility is less than or equal to the upper bound h.

2.2. Radial approach to min-max optimization

The problem (1) – (8) is also known as the p-centre problem, which is the task of determining the maximum p network nodes as service centre locations so that the maximal disutility perceived by the worst situated user is minimal. Nevertheless, the p-centre problems associated with the above-mentioned service system design are characterised by a considerably large number of possible service centre locations. The location-allocation model constitutes this mathematical programming problem, which resists any attempt at a fast solution. A similar situation arises when large instances of the p-median problem were solved. At that time, it was found that large instances of the covering problem could be easily resolved using common optimization software. The necessity of solving large instances of the p-median problem has led to radial formulation [1, 4, 5 and 6]. This approach avoids assigning the individual user location to some of the located service centres and deals only with the information, regardless of whether a particular service centre is located within a given user radius. The later approach leads to the model that is similar to the set covering problem, and can be easily solved even for large instances using common optimisation software. To model a decision on locating a service centre at a particular location i, the zero-one variable $y_i \in \{0, 1\}$ was used as in the previous subsection. In the same sense, the variable h was also applied, i.e. as the upper bound of all perceived disutility values. To obtain an upper or a lower bound of the original objective function, the range $[d_0, d_m]$ of all possible $m+1$ disutility values $d_0 < d_1 < \dots < d_m$ from the matrix $\{d_{ij}\}$ is partitioned into $v+1$ zones according to [7, 8]. The zones are separated by a finite ascending sequence of so-called *dividing points* $D_1, D_2 \dots D_v$ chosen from the sequence $d_0 < d_1 < \dots < d_m$, where $0 = d_0 = D_0 < D_1$ and also $D_v < D_{v+1} = d_m$. The zone s corresponds to the interval $(D_s, D_{s+1}]$. The length of the s-th interval is denoted by e_s for $s = 0 \dots v$. Further, auxiliary zero-one variables x_{js} for $s = 0 \dots v$ are introduced. The variable x_{js} takes the value of 1, if the disutility of the user at $j \in J$ from the nearest located centre is greater than D_s, or takes the value of 0 otherwise. Then, the expression $e_0x_{j0} + e_1x_{j1} + \dots + e_vx_{jv}$ constitutes an upper approximation of disutility d_{j*} from the user location j to the nearest located service centre. If disutility d_{j*} belongs to the interval $(D_s, D_{s+1}]$, then the value of D_{s+1} is the upper estimation of d_{j*} with a maximal possible deviation e_s. This requires introducing a zero-one constant a_{ij}^s for each triple $[i, j, s]$, where $i \in I$, $j \in J$, $s \in [0 \dots v]$. The constant a_{ij}^s is equal to 1, if disutility d_{ij} between the user location j and the possible centre location i is less than or equal to D_s, otherwise a_{ij}^s is equal to 0. Then the radial min-max public service system design problem can be formulated as follows.

$$Minimize \quad h \tag{9}$$

$$Subject\ to: \quad x_{js} + \sum_{i \in I} a_{ij}^s y_i \geq 1 \quad for\ j \in J,\ s = 0,1 \dots v \tag{10}$$

$$\sum_{i \in I} y_i \leq p \tag{11}$$

$$\sum_{s=0}^{v} e_s x_{js} \leq h \quad for\ j \in J \tag{12}$$

$$y_i \in \{0,1\} \quad for\ i \in I \tag{13}$$

$$x_{js} \geq 0 \quad for\ j \in J,\ s = 0,1 \dots v \tag{14}$$

$$h \geq 0 \tag{15}$$

In this model, the objective function (9) defined by the single variable h gives the upper bound of all perceived disutility values. The constraints (10) ensure that the variables x_{js} are allowed to take the value of 0, if there is at least one centre located in radius D_s from the user location j and the constraint (11) limits the number of located service centres by p. The link-up constraints (12) ensure that each perceived disutility is less than or equal to the upper bound h.

2.3. Bisection radial approach to min-max optimization

The bisection radial approach makes use of the radial model, but uses only its reduced form in determining whether there is a solution with the objective function value less than or equal to the given disutility value D_s. In this model, the zero-one variables $y_i \in \{0, 1\}$ for $i \in I$ are also used to model the decision on locating a service centre at the location i. The variables x_j are introduced to indicate whether the disutility of the users at the location $j \in J$ following from the nearest located centre is greater than D_s. In such a case, the variable takes the value of 1, or 0 otherwise. The corresponding model is formulated as follows.

$$\textit{Minimize} \quad \sum_{j\in J} x_j \tag{16}$$

$$\textit{Subject to:} \quad x_j + \sum_{i\in I} a_{ij}^s y_i \geq 1 \quad \textit{for } j \in J \tag{17}$$

$$\sum_{i\in I} y_i \leq p \tag{18}$$

$$y_i \in \{0,1\} \quad \textit{for } i \in I \tag{19}$$

$$x_j \geq 0 \quad \textit{for } j \in J \tag{20}$$

In this model, the objective function (16) represents the number of user locations, where the perceived disutility is greater than D_s. The constraints (17) ensure that variables x_j are allowed to take the value of 0, if there is at least one centre located within radius D_s extending from the user location j and constraint (18) limits the number of located service centres by p.

With the reduced form of the radial approach (16) – (20), the dividing points are not needed, because the smallest relevant disutility DmM is searched by the bisection method that is applied on the whole disutility range. This allows the exact solution of the former problem (1) – (8) to be obtained by iterative solution to the radial model (16) – (20), thus enabling to take advantage of the set-covering problem.

3. Computational study

3.1. Research goals

The main goal of this study is to develop and verify an effective method for solving the problem associated with the min-max optimal public service system design, and involves identifying the smallest relevant disutility value *DmM* used in the lexicographic optimisation process. The previous experiments described in [11] indicated that this important first step is the most time-consuming part of the whole algorithm. In the previous section, three possible approaches to this problem were suggested: original location-allocation approach, radial approach based on radial formulation of the p-median problem and the bisection radial approach based on reduced radial formulation accompanied by the bisection method. This paper endeavours to answer the question of whether the radial approach considerably accelerates the algorithm of the solving the p-centre problem. Therefore, we compare the location-allocation approach based on the formulation (1) – (8) with the radial approach based on the covering model (9) – (15). Then, we explore the properties of the bisection radial approach and compare it to the previous two approaches. When using the radial model (9) – (15), the set of optimal dividing points defining particular zones was determined by the procedure described in [9, 10].

3.2. Benchmarks and used optimisation software

All reported experiments were performed using the optimisation software FICO Xpress 7.5 (64-bit, release 2013) for both the location-allocation model and the radial approaches. The associated code was run on a PC equipped with the Intel® Core™ i7 2630QM processor running at 2.0 GHz and 8 GB RAM.

Particular approaches were tested on the pool of benchmarks obtained from the road network of the Slovak Republic. The instances were organised so that they corresponded to the administrative organisation of Slovakia. A corresponding number of inhabitants b_j was taken for each city and settlement in particular self-governing region (Bratislava - BA, Banská Bystrica - BB, Košice - KE, Nitra - NR, Prešov - PO, Trenčín - TN, Trnava - TT and Žilina - ZA). The coefficients b_j were rounded to the hundred. The number of possible service centre locations $|I|$ was the same as the number of user locations $|J|$ in all solved instances. That meant that each community (even the smallest) could represent a possible service centre location. The network distance from a user to the nearest located centre was taken as a user disutility. To obtain a bigger pool of benchmarks for the computational study, the value of p was set in such a way, so that the ratio of $|I|$ to p equalled 2, 3, 4, 5, 10, 15, 20, 30, 40, 50 and 60 respectively. The number m+1 of all possible disutility values in the sequence d_0, d_1 ... d_m depended on the parameter p which limited the number of located service centres. As shown in [7], when minimizing the objective function value of the radial model, the p-1 largest disutility value from each matrix column can be excluded as non-relevant. Thus, the value of m+1 may differ for each benchmark even for the same self-governing region. The size of the set I together with the value of parameter p and the associated value of m+1 for each self-governing region have been entered in Table 1 below. The associated results of numerical experiments are reported in the following subsection.

Region	$\|I\|$		Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8	Case 9	Case 10	Case 11
BA	87	p	44	29	22	18	15	9	6	5	3	2	---
		m+1	55	62	64	65	66	68	71	72	75	77	---
BB	515	p	258	172	129	103	52	35	26	18	13	11	9
		m+1	107	131	148	155	166	171	173	175	176	177	177
KE	460	p	230	154	115	92	46	31	23	16	12	10	8
		m+1	118	156	165	171	182	185	187	189	192	193	194
NR	350	p	175	117	88	70	35	24	18	12	9	7	6
		m+1	88	97	102	108	118	121	124	126	128	130	130
PO	664	p	332	222	166	133	67	45	34	23	17	14	12
		m+1	135	157	168	180	215	223	226	229	231	233	235
TN	276	p	138	92	69	56	28	19	14	10	7	6	5
		m+1	105	117	125	128	134	137	139	141	142	143	144
TT	249	p	125	83	63	50	25	17	13	9	7	5	---
		m+1	98	114	123	129	141	146	149	151	152	154	---
ZA	315	p	158	105	79	63	32	21	16	11	8	7	6
		m+1	101	115	123	128	136	139	141	143	145	148	149

Table 1. *Size of tested benchmarks*

3.3. Results of numerical experiments

The results for the region of Žilina covering all generated instances, which differ for parameter p, have been entered in Table 2. The first two columns are necessary for identifying the benchmark. As was mentioned previously, parameter p limited the number of located service centres with the value of $m+1$ representing the number of all possible disutility values in the sequence d_0, d_1 ... d_m, where only the $|I|$ - p +1 smallest values from each matrix column were included. The experiments were organised so that each benchmark was solved using all studied approaches. Computational times in seconds for particular method are given in the column denoted by *Time* and the maximum relevant disutility is given in the *DmM* columns. Here, it is important to note that the radial approach described by the model (9) − (15) consists of two optimisation processes. First, the optimal set of dividing points is computed then the radial model (9) − (15) is solved. Values reported in Table 2 represent the global time for both processes. The bisection radial approach does not require the dividing points, but the computational process is formed by solving of the radial model (16) − (20) in an iterative manner for different disutility values D_s as defined by the bisection method applied on the whole disutility range. Thus, the reported computational time contains all iterations. The total number of performed iterations is given in the column denoted by *NoI*.

p	m+1	Location-allocation approach (1) – (8)		Radial approach (9) – (15)		Bisection radial approach (16) – (20)		
		Time [s]	*DmM*	Time [s]	*DmM*	Time [s]	*DmM*	*NoI*
158	101	16.34	4	6.06	4	0.13	4	7
105	115	39.12	6	5.12	6	0.14	6	7
79	123	38.92	7	4.89	7	0.15	7	7
63	128	60.45	8	5.78	8	0.29	8	7
32	136	66.52	14	3.72	14	0.26	14	8
21	139	78.21	16	9.24	16	0.24	16	8
16	141	61.08	20	9.68	21	0.26	20	8
11	143	68.13	25	13.62	26	0.69	25	8
8	145	85.19	31	28.06	31	0.33	31	8
7	148	156.01	34	26.37	36	0.35	34	8
6	149	154.00	34	29.38	36	0.35	34	8

Table 2: *Results of the experiments for the self-governing region of Žilina with $|I| = 315$ possible service centre locations*

Since the detailed results for other self-governing regions had similar characteristics as was obtained for the region of Žilina, only selected instances for the other regions have been reported. The value of parameter p in these instances was chosen to correspond to the original set of problems from a real-life emergency medical service system (the ratio of $|I|$ to p takes the value around 10). In the following Table 3 the same denotation as before is used.

Region	$\|I\|$	p	m+1	Location-allocation approach (1) – (8)		Radial approach (9) – (15)		Bisection radial approach (16) – (20)		
				Time [s]	*DmM*	Time [s]	*DmM*	Time [s]	*DmM*	*NoI*
BA	87	9	68	1.47	14	0.65	14	0.03	14	7
BB	515	52	166	496.01	13	6.77	13	0.41	13	8
KE	460	46	182	195.98	12	10.48	12	0.67	12	8
NR	350	35	118	112.10	13	5.02	13	0.72	13	7
PO	664	67	215	1042.41	12	17.57	12	2.82	12	8
TN	276	28	134	29.74	12	3.50	12	0.22	12	8
TT	249	25	141	68.65	13	3.54	13	0.38	13	8
ZA	315	32	136	66.52	14	3.72	14	0.26	14	8

Table 3: *Results of the experiments for the self-governing regions of Slovakia for the selected parameter p*

4. Conclusions

The main goal of this study was to introduce and compare three different approaches to solving the problem of the min-max optimal public service system design for the initial step of the lexicographic optimisation process. We have solved several real instances and compared suggested models.

Based on the reported results, the location-allocation approach proved to be the most demanding as concerns computational time. Therefore, this approach was not suitable for large instances due to its complexity and memory requirements.
Simple reformulation of the location-allocation model into the radial form did not result in considerable improvement. Despite some reduction of computational time, the results were not convincing as had been expected. We presume that the link-up constraints of the upper bound definition degraded the useful features of the radial model, which proved its effectiveness in solving the min-sum location problem. Furthermore, the accuracy of the result strongly depends on disutility approximation by implementing dividing points. In some cases, this method provided such a solution, which differed considerably from the optimal one.

The third approach combining the reduced radial model with the bisection method applied to the whole disutility range proved to be the most suitable. Its main advantage is providing an exact solution and a small computational time. It is in two orders faster in comparison to the former location-allocation approach and the respective model is much smaller.

Thus, we can conclude that we have constructed a very useful tool for the solving the problem of a middle-sized min-max optimal public service system design, which can be implemented simply using readily available commercial optimisation software. The initial phase studied and successfully solved in this paper plays a very important role in not only the lexicographic optimisation process, but it also provides considerable reduction of the set of effective disutility values, thus saving a lot of computational time.

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