The Nonlinear Generalized Transportation Problem with convex costs*

Marcin Anholcer[1,†]

[1] *Poznań University of Economics, Faculty of Informatics and Electronic Economy, Al. Niepodległości 10, 61-875 Poznań, Poland*
E-mail: ⟨*m.anholcer@ue.poznan.pl*⟩

Abstract. We investigate the Nonlinear Generalized Transportation Problem (NGTP), where the transportation costs and the costs that depend on the amount of good delivered to the destination points are strictly convex functions. The amounts of goods change during the transportation process. This model may correspond, for example, with a congested network where the time costs are involved. First, we present the NGTIP model, then provide a method of solving the problems of this type and then we prove convergence of the method. The numerical experiments prove the effectiveness of the presented algorithm.

Key words: Nonlinear Transportation Problem, Nonlinear Generalized Transportation Problem, Stochastic Transportation Problem, Stochastic Generalized Transportation Problem, Equalization Method.

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1. Introduction

The Generalized Transportation Problem (GTP) is a version of the Transportation Problem treating the likelihood that the amounts of goods transported from supply points to destinations change during the transportation process (in particular, if the amount decreases, then the change is represented by a reduction ratio). It is special case of the Generalized Minimum Cost Flow Problem, described by Ahuja et al. ([1], Chapter 15). The authors also provide some examples of applications of generalized flows. A polynomial algorithm for this problem was proposed by Wayne in [51]. Goldberg et al. presented combinatorial algorithms for the Generalized Circulation Problem in [19]. The generalized network flows were also discussed by Cohen and Megiddo [13] and Glover et al. in [18].

The Generalized Transportation Problem was discussed by Balas [8], Balas and Ivanescu [9], Gupta [20], Lourie [30], MacKinnon [31] and Rowse [43]. Anholcer and Kawa in [7] analyzed the relation between the reduction ratio and the structure of the optimal solution. Some variants of the problem were also discussed in [22].

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†Corresponding author.

In the nonlinear variants of GTP, the objective function is not linear. In this paper we assume that the unit transportation costs are not necessarily constant and, moreover, there are additional costs assigned to every destination point, depending only on the total amount of goods delivered to this point. A special case is the Stochastic Generalized Transportation Problem (SGTP) where unit transportation costs are constant. This problem was studied by Qi in [37], where the author analyzed the generalized version of his method presented in [36]. The Stochastic Transportation Problem was studied by Williams in [52]. The special case of the Stochastic Transportation Problem with single sourcing was studied by Romeijn and Sargut [39]. Sikora [44] analyzed the case when the demand distributions are discrete. The variants of Stochastic Transportation Problem were studied also by Szwarc [47].

Various variants of the Stochastic Transportation Problem were also studied by Holmberg [23], Holmberg and Jörnsten [24] and Holmberg and Tuy [25]. A survey of nonlinear continuous allocation problems, including transportation problems are found in Patriksson's paper [38]. The nonlinear versions of the Transportatioin Problem are also discussed in [50, pp.100–125].

The three-dimensional transportation problem was considered by Moantain [33]. A nonlinear version of the Transportation Problem with a fixed number of sources was studied by Cosares and Hochbaum in [14]. Other approaches to the nonlinear variants of the Transportation Problem (including heuristics) were discussed by Cao and Uebe in [12], Dangalchev in [15], Jo et al. in [27] (see also the comments to this paper published by Kannan et al. in [28]), Ilich and Simonovic in [26], Klanšek and Pšunder in [29], Tuy et al. in [48] and [49].

In this article we are particularly interested in the Transportation Problem characterized by convex transportation costs. This type of cost functions appears, for example, when a congested network is taken into consideration and the transportation costs involve also the cost of time. This kind of situation has been discussed in detail by Ahuja, Magnanti and Orlin in [1] (p.547, Urban Traffic Flows). The convex transportation costs in congested networks have been studied by Mandayan and Prabhakar [32], Roughgarden [40] and Roughgarden and Tardos [41], [42]. The convex congestion costs in the context of transportation-location problems have been considered by Descrochers, Marcotte and Stan [16], and Harkness and ReVelle [21]. Other examples of problems associated with the convexity assumption can be found in papers published by Dobre [17], and Nguyen and Tan [35]. Monma and Segal in [34] describe three applications of convex cost network models in communication systems. Other examples of problems expressed as a Transportation Problem with convex costs have been discussed in [1] (pp.547-551: Area Transfers in Communication Networks, Matrix Balancing, the Stick Percolation Problem). Three other examples of problems that are transformable to convex network flow problems are listed by Cheng in [11]. These are water distribution, electrical network analysis and equilibrium export-import trade problems. Another problem that can be considered is that which is transformed to a form of the convex Transportation Problem (in fact it is a case of the Matrix Balancing Problem). In certain sales systems (for example in the case of household appliances), the management of the producer prepares the sales plans for coming year. The plan includes aggregates, i.e. the amount (or value) of product sales, and the total value of sales to customers (i.e., shops and warehouses). The goal of the sales department is to plan product sales to customers so that all side constraints are satisfied. The detailed sales plan has to be as similar as possible to historic sales if it is to be realistic. One of the popular measures of the closeness of two matrices is the Euclidean distance between them. Thus, the Euclidean distance between the planned sales matrix and the historical one is minimized. If we assign the product plans to the supply points, the customer plans to the destination points and the flows to the sales plans for the respective product-customer pairs, then we obtain an instance of the Transportation Problem with quadratic, convex transportation cost functions.

The very early version of the Equalization Method was presented by Sikora in [45] and by Sikora et al. in [46]. It was designed to solve transportation problems without demand limits and with additional quadratic, convex costs assigned to the destination points. The generalization of the method for all types of convex functions was presented by Anholcer (see [2, 3, 4]) and the convergence of the algorithm was proved (see [2, 3]). A special version of the Equalization Method for the Stochastic Generalized Transportation Problem has been presented by Anholcer in [5].

The method of progressive analysis of variables for the Stochastic Transportation Problem with discrete demand distribution was presented by Sikora in [44]. The Stochastic Generalized Transportation Problem with discrete demand distribution has been studied by Anholcer in [6].

The article is organized as follows. In the next section, we formulate the problem and the optimality conditions. In Section 3, we present the algorithm. Convergence issues are discussed in Section 4. An illustrative example is presented in Section 5. 6 Computatinal experiments are discussed in Section 6. The paper finishes with some remarks and suggestions for further work. According to the author's knowledge, up till now no research on NGTP has been performed, hence the content in following sections is considered a unique contribution to this particular field.

2. Formulating the problem

In the ordinary Generalized Transportation Problem, a uniform good is transported from m supply points to n destinations. During the transportation process, the amount of good changes. In most practical applications it decreases. Let x_{ij} be the amount of good sent from supply point i to destination j. Then the amount of good that reaches destination j equals to $r_{ij}x_{ij}$, where r_{ij} denotes the so-called *multiplier*. The unit transportation cost c_{ij} from each supply point $i = 1, \ldots, m$ to each destination $j = 1, \ldots, n$ is constant, the demand b_j of every destination has to be satisfied and for every supply point i, its supply cannot be exceeded. It follows that the model has the following form

$$\begin{aligned}
&\min \left\{ f(x) = \textstyle\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \right\} \\
&s.t. \\
&\textstyle\sum_{i=1}^m r_{ij} x_{ij} = b_j, \text{ for } j = 1, \dots, n, \\
&\textstyle\sum_{j=1}^n x_{ij} \le a_i \quad \text{ for } i = 1, \dots, m, \\
&x_{ij} \ge 0, \quad\quad\quad \text{ for } i = 1, \dots, m, j = 1, \dots, n.
\end{aligned}$$

The assumption about linearity of costs is, however, not realistic. First, the transportation costs do not have to be linear, as the examples from the Introduction show. Indeed, in the practical applications, the cost c_{ij} very often has the form of a nonlinear function in x_{ij}, increasing and convex for $x_{ij} \ge 0$. On the other hand, some additional costs may appear at destination points. In particular this may be the costs of transforming the transported good at the destination points or the distribution and promotion costs. In the case of non-deterministic demand, one may include the expectation of the shortage and surplus costs. In those cases we can assume that the cost functions are convex. Thus we assume that an increasing convex function is assigned to every destination point j. Observe that this implies that the objective function is convex. The limitation b_j on the amount of good delivered to every destination j is not included in the list of constraints, i.e. we assume that the demand is not limited in a short time horizon, which is also a realistic assumption for many goods (in particular innovative products having just been introduced to the market). This is the motivation to consider the following nonlinear variant of GTP. In the remainder of the paper, we will refer to this problem as the Nonlinear Generalized Transportation Problem (NGTP).

$$\begin{aligned}
&\min \left\{ f(x) = \textstyle\sum_{i=1}^m \sum_{j=1}^n c_{ij}(x_{ij}) + \sum_{j=1}^n f_j(x_j) \right\} \\
&s.t. \\
&\textstyle\sum_{i=1}^m r_{ij} x_{ij} = x_j, \text{ for } j = 1, \dots, n, \\
&\textstyle\sum_{j=1}^n x_{ij} \le a_i \quad \text{ for } i = 1, \dots, m, \\
&x_{ij} \ge 0, \quad\quad\quad \text{ for } i = 1, \dots, m, j = 1, \dots, n.
\end{aligned}$$

Here $c_{ij}(x_{ij})$ are increasing, convex functions in x_{ij} and $f_j(x_j)$ are convex functions in x_j, where x_j denotes the total amount of good delivered to destination j. We assume that each of the functions c_{ij} and f_j is differentiable in every point of its domain. Given a function $f(x)$, we will denote its derivative with $f'(x)$.

We can derive the following KKT optimality conditions for NGTP.

Theorem 2.1. *A solution $x = \{x_{ij}, x_j | i = 1, \dots, m, j = 1, \dots, n\}$ of the above problem is the global optimum if and only if there exist u_i, $i = 1, \dots, m$ such that for every $i = 1, \dots, m$ $j = 1, \dots, n$*

$$\begin{aligned}
c'_{ij}(x_{ij}) + r_{ij} f'_j(x_j) \ge u_i, \ &\textit{if } x_{ij} = 0, \\
c'_{ij}(x_{ij}) + r_{ij} f'_j(x_j) = u_i, \ &\textit{if } x_{ij} > 0.
\end{aligned}$$

Proof. Using duality theory and some basic transformations, we observe immediately that the listed conditions are equivalent to the KKT conditions for the mentioned problem. As the set of feasible solutions is a convex polytope and the objective function is convex, the KKT conditions are both necessary and sufficient conditions (see e.g.[10, p. 207]). □

In the next section we are going to present the algorithm that converges to the point satisfying the above conditions.

3. The algorithm

Let us introduce additional variables $x_{i,n+1}$, $i = 1, \dots, m$ and x_{n+1}. Then we can write NGTP in the following equivalent form

$$\begin{array}{ll} \min \left\{ f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n+1} c_{ij}(x_{ij}) + \sum_{j=1}^{n+1} f_j(x_j) \right\} & \\ s.t. & \\ \sum_{i=1}^{m} r_{ij} x_{ij} = x_j, & \text{for } j = 1, \dots, n+1, \\ \sum_{j=1}^{n+1} x_{ij} = a_i & \text{for } i = 1, \dots, m, \\ x_{ij} \geq 0, & \text{for } i = 1, \dots, m, j = 1, \dots, n+1. \end{array}$$

Here $c_{i,n+1}(x_{i,n+1})$ and $f_{n+1}(x_{n+1})$ are equal to 0 everywhere. The following algorithm solves the last problem, so also NGTP.

Algorithm 1 Equalization method for NGTP

Step 1: *Finding the initial solution.* Assume an accuracy level $\varepsilon > 0$. Let the first solution be defined by the formulae

$$\begin{array}{l} x_{ij} = a_i,\ j = n+1, \\ x_{ij} = 0,\ \ j < n+1. \end{array} \tag{1}$$

Calculate the initial values of the derivatives:

$$\begin{array}{ll} k_{ij} = c'_{ij}(0) + r_{ij} f'_j(0), & i = 1, \dots, m, j = 1, \dots, n, \\ k_{i,n+1} = 0, & i = 1, \dots, m. \end{array} \tag{2}$$

Proceed to step 2.

Step 2: *Checking the optimality.* For every i calculate

$$v_i = \min\{k_{ij} | j = 1, \dots, n+1\} \tag{3}$$

and

$$w_i = \max\{k_{ij} | x_{ij} > 0, j = 1, \dots, n+1\} - v_i. \tag{4}$$

If $\max\{w_i | i = 1, \dots, m\} < \varepsilon$, then STOP, the assumed accuracy level has been reached. Otherwise go to step 3.

Let us briefly discuss the steps of the algorithm. Imagine that we are using a 2-dimensional tableau, with supply points in rows and the destinations in columns.

Algorithm 1 Equalization method for NGTP (continued)

Step 3: *Transforming the solution.* Choose the smallest $i^\star$ such that

$$w_{i^\star} = \max\{w_i | i = 1, \dots, m\}. \tag{5}$$

Choose the smallest $j^\star$ and $j^{\star\star}$ such that

$$\begin{aligned} k_{i^\star j^\star} &= \max\{k_{i^\star j} | x_{i^\star j} > 0, j = 1, \dots, n+1\}, \\ k_{i^\star j^{\star\star}} &= \min\{k_{i^\star j} | j = 1, \dots, n+1\}. \end{aligned} \tag{6}$$

Let

$$\begin{aligned} \delta^-(\lambda) &= r_{i^\star j^\star} f'_{j^\star}(x_{j^\star}) + c'_{i^\star j^\star}(x_{i^\star j^\star}) + \\ &\quad - r_{i^\star j^\star} f'_{j^\star}(x_{j^\star} - r_{i^\star j^\star}\lambda) - c'_{i^\star j^\star}(x_{i^\star j^\star} - \lambda), \\ \delta^+(\lambda) &= r_{i^\star j^{\star\star}} f'_{j^{\star\star}}(x_{j^{\star\star}} + r_{i^\star j^{\star\star}}\lambda) + c'_{i^\star j^{\star\star}}(x_{i^\star j^{\star\star}} + \lambda) + \\ &\quad - r_{i^\star j^{\star\star}} f'_{j^{\star\star}}(x_{j^{\star\star}}) - c'_{i^\star j^{\star\star}}(x_{i^\star j^{\star\star}}) \end{aligned} \tag{7}$$

Let $\lambda^\star$ be the smallest solution of the equation

$$\delta^-(\lambda) + \delta^+(\lambda) = w_{i^\star}. \tag{8}$$

If $\lambda^\star > x_{i^\star j^\star}$, then set

$$\lambda^\star \leftarrow x_{i^\star j^\star}. \tag{9}$$

Set

$$\begin{aligned} & x_{i^\star j^\star} \leftarrow x_{i^\star j^\star} - \lambda^\star, \\ & x_{i^\star j^{\star\star}} \leftarrow x_{i^\star j^{\star\star}} + \lambda^\star, \\ & x_{j^\star} \leftarrow x_{j^\star} - r_{i^\star j^\star}\lambda^\star, \\ & x_{j^{\star\star}} \leftarrow x_{j^{\star\star}} + r_{i^\star j^{\star\star}}\lambda^\star, \\ & k_{ij} \leftarrow c'_{ij}(x_{ij}) + r_{ij} f'_j(x_j), i = 1, \dots, m, j \in \{j^\star, j^{\star\star}\}, \end{aligned} \tag{10}$$

and go back to step 2.

In step 1, all the flows are placed in the last column and the initial values of partial derivatives are computed.

In step 2, the optimality indicators are computed. To be more specific, v_i and w_i correspond to the left sides of the KKT conditions.

In step 3, for each row, the relative violation of the respective KKT condition is calculated. Then the row with highest violation is chosen, as well as the columns that caused this violation. This is equivalent to choosing the cells of the tableau with the lowest and highest partial derivative in the chosen row. Finally, the flow is moved from the cell with the highest derivative to the cell with the lowest derivative so that either the entire flow is moved or the respective derivatives equate in the next step.

It is evident that the value of the objective decreases strictly in each step.

4. Convergence of the method

Let A denote the algorithmic map of the above algorithm. Let B be the map finding the search direction and let C be the map finding the next feasible solution when the search direction is already given. Let X denote the set of feasible solutions of the NGTP problem, and let D denote the set of search directions. X is a compact set, as it is closed and bounded. D is compact as it is finite. Indeed, for set m and n, there are m possible choices of $i^\star$ and $2\binom{n+1}{2}$ choices of the pair $(j^\star, j^{\star\star})$. As this determines the positions of 1 and -1, the only non-zero elements of any vector in D, we have $|D| = 2m\binom{n+1}{2}$.

Now we are going to prove that both mappings B and C are closed. Let us recall, that the algorithmic map $A : X \to Y$ is closed at $x \in X$, if for any sequences $\{x^k\}$ and $\{y^k\}$ (k denotes the number of iteration) such that $x^k \in X$, $x^k \to x$, $y^k \in A(x^k)$, $y^k \to y$, we have that $y \in A(x)$ (see e.g. [10, p.321]).

Lemma 4.1. *The algorithmic map $B : X \to D$ in the Equalization Method for NGTP is closed on X.*

Proof. Assume that $x^k \in X$, $x^k \to x$, $d^k \in B(x^k)$, $d^k \to d$. It is enough to show that $d \in B(x)$.

As D is finite, $d^k = d$ must hold for almost all k. It implies that starting from some iteration k, we have that $d^k \in B(x^k)$ and $d^k \to d$ are equivalent to $d \in B(x^k)$. Let us choose an $\varepsilon > 0$.

Let k_1 be the value of k, starting from which all the elements of $\{d^k\}$ are equal to d. Let k_2 be the value of k, starting from which the inequality $||x^k - x|| < \varepsilon$ holds (such a value exists, as $x^k \to x$). Let $k_0 = \max\{k_1, k_2\}$. Then for $k > k_0$ we have

$$|x_{ij}^k - x_{ij}| < \varepsilon, i = 1, \ldots, m, j = 1, \ldots, n+1.$$

As all the derivatives are finite in the domain, the Lipschitz condition is fulfilled, i.e.

$$|k_{ij}^k - k_{ij}| < L\varepsilon, i = 1, \ldots, m, j = 1, \ldots, n+1,$$

where L is a constant (depending only on the input data). It follows that

$$\begin{aligned} &|w_i^k - w_i| < 2L\varepsilon, i = 1, \ldots, m, \\ &|\max\{w_i^k | i = 1, \ldots, m\} - \max\{w_i | i = 1, \ldots, m\}| < 4L\varepsilon, \end{aligned}$$

which guarantees the same choice of $i^\star$ for both solutions, provided that ε is sufficiently small. Having chosen the same $i^\star$, we obtain

$$\begin{aligned} &|\max\{k_{i^\star j}^k | x_{i^\star j}^k > 0, j = 1, \ldots, n+1\} - \max\{k_{i^\star j} | x_{i^\star j} > 0, j = 1, \ldots, n+1\}| < 2L\varepsilon, \\ &|\min\{k_{i^\star j}^k | j = 1, \ldots, n+1\} - \min\{k_{i^\star j} | j = 1, \ldots, n+1\}| < 2L\varepsilon, \end{aligned}$$

which guarantees the same choice of $j^\star$ and $j^{\star\star}$ for both solutions, provided that ε is sufficiently small. As we have already observed, $d \in B(x^k)$. As the unique choice of $i^\star$, $j^\star$ and $j^{\star\star}$ uniquely defines the search direction d, we obtain that $B(x^k) = B(x)$ and consequently $d \in B(x)$, as desired. $\square$

Lemma 4.2. *The algorithmic map $C : D \to X$ in the Equalization Method for NGTP is closed on D.*

Proof. Assume that $d^k \in D$, $d^k \to d$, $x^k \in C(d^k)$, $x^k \to x$. It is enough to show that $x \in C(d)$.

Similarly as in the previous proof we observe that for almost all k we have $d^k = d$, and so $x^k \in C(d)$. As for the set d, there is a unique choice of x, and it follows that $|C(d)| = 1$ and we can write $x^k = C(d)$. As we also have $x^k \to x$, it follows that $x = C(d)$ ($x \in C(d)$), as desired. □

We will use the following fact.

Lemma 4.3. *[10, p.325,Corollary 1] Let $X \subset R^p$, $Y \subset R^q$ and $Z \subset R^r$ be nonempty closed sets. Let $B : X \to Y$ and $C : Y \to Z$ be point-to-set maps. Suppose that B is closed at x, C is closed on $B(x)$ and Y is compact. Then $A = CB$ is closed at x.*

>From the above three lemmas it follows that

Lemma 4.4. *The algorithmic map $A : X \to X$ in the Equalization Method for NGTP is closed on X.*

Proof. Put $Y = D$ and $Z = X$ in the lemma 4.5. As B and C are closed and D is compact, the result follows. □

In order to finish the main theorem, we need one more general result.

Lemma 4.5. *[10, p.321,Theorem 7.2.3] Let $X \subset R^n$ be a nonempty closed set and $\Omega \subseteq X$ a nonempty solution set. Let $A : X \to X$ be a point-to-set map. Given $x^1 \in X$, the sequence $\{x^k\}$ is generated iteratively as follows: If $x^k \in \Omega$, then STOP, otherwise let $x^{k+1} \in A(x^k)$, set $k \leftarrow k+1$ and repeat. Suppose that the sequence $\{x^k\}$ is contained in a compact subset of X, and assume that there is a continuous function $\alpha : R^n \to R$ such that $\alpha(y) < \alpha(x)$ if $x \notin \Omega$ and $y \in A(x)$. If the map A is closed over the complement of Ω, then either the algorithm stops in a finite number of steps or it generates an infinite sequence $\{x^k\}$ such that all accumulation points of $\{x^k\}$ belong to Ω and $\alpha(x^k) \to \alpha(x)$ for some $x \in \Omega$.*

This allows us to formulate the following convergence theorem.

Theorem 4.6. *The Equalization Method for NGTP is convergent.*

Proof. The set
$$\Omega = \{x \,| \max\{w_i | i = 1, \dots, m\} < \varepsilon\}$$
is nonempty as the NGTP has at least one optimal solution, where
$$\max\{w_i | i = 1, \dots, m\} = 0.$$
The sequence $\{x^k\}$ is contained entirely in X, which is compact. If we define α as the objective function, then we obtain that $\alpha(y) < \alpha(x)$ for all $y \in A(x)$, $x, y \in X$. From the last lemma it follows that the Equalization Method either stops after finite number of steps, or each of the accumulation points of $\{x^k\}$ belongs to Ω. □

5. Illustrative example

In order to make the algorithm easier to understand, let us analyze the following simple example. Assume there are two supply points and three destination points. Let $a_1 = 10$, $a_2 = 12$, $c_{11}(x_{11}) = 0.1x_{11}^2$, $c_{12}(x_{12}) = 0.15x_{12}^2$, $c_{13}(x_{13}) = 0.25x_{13}^2$, $c_{21}(x_{21}) = 0.05x_{21}^2$, $c_{22}(x_{22}) = 0.2x_{22}^2$, $c_{23}(x_{23}) = 0.35x_{23}^2$, $f_1(x_1) = 1/4x_1^2 - 4x_1 + 20$, $f_2(x_2) = 5/12x_2^2 - 6x_2 + 36$, $f_3(x_3) = 15/28x_3^2 - 10x_3 + 70$, $r_{11} = 0.9$, $r_{12} = 0.8$, $r_{13} = 0.7$, $r_{21} = 0.8$, $r_{22} = 0.9$ and $r_{23} = 0.8$. The results of the iterations have been presented in tabular form. For each iteration, the values of flows x_{ij}, the derivatives $\frac{\partial f}{\partial x_{ij}} = k_{ij}$, the dual variables v_i, the optimality indicators w_i, the total cost $f(x)$, accuracy α and step length λ have been presented in the following tables. The stopping criterion assume that $\alpha < \varepsilon = 0.1$. Recall that in order to solve the problem we introduce additional destination with slack variables (here for $j = 4$), and setting $c_{14}(x_{14}) = 0$, $c_{24}(x_{24}) = 0$, $f_4(x_4) = 0$ and $r_{14} = r_{24} = 1$.

x_{ij}	$j=1$	$j=2$	$j^{\star\star}=3$	$j^{\star}=4$	$f(x)$	126
$i=1$	0	0	0	10	α	8
$i^{\star}=2$	0	0	0	12	λ	5.7732
k_{ij}	$j=1$	$j=2$	$j^{\star\star}=3$	$j^{\star}=4$	v_i	w_i
$i=1$	-3.6	-4.8	-7	0	-7	7
$i^{\star}=2$	-3.2	-5.4	-8	0	-8	8

Table 1: *Numerical example - Initialization*

Let us start with initialization (Table 1). The first solution is computed with the formulae (1): $x_{14} = a_1 = 10$, $x_{24} = a_2 = 12$, and $x_{ij} = 0$ for $j < 4$. The derivatives are calculated with the formulae (2). For example, for $i = 1$ and $j = 2$ we have $c'_{12}(x_{12}) = 0.3x_{12}$ and $f'_2(x_2) = 5/6x_2 - 6$, so $k_{12} = c'_{12}(0) + r_{12}f'_2(0) = 0 + 0.8 \cdot (-6) = -4.8$. For $i = 2$ and $i = 4$, we have $k_{24} = 0$. Now, according to the formula (3), u_i is equal to the minimum k_{ij} in the row, so $v_1 = -7$ and $v_2 = -8$. As in each row the maximum k_{ij} over the pairs (i, j) is such that $x_{ij} > 0$ are 0 (only $x_{14} > 0$ and $x_{24} > 0$), using the formula (4) we obtain $w_1 = 0 - (-7) = 7$ and $w_2 = 0 - (-8) = 8$. This means that the accuracy $\alpha = \max\{w_i | i = 1, \ldots, m\} > \epsilon$ and so we continue. The highest w_i is w_2, so according to the formula (5), $i^\star = 2$. Now, when i is set, using the formulae (6) we find $j^\star = 4$ and $j^{\star\star} = 3$. Having established the search direction, we use the formulae (7) in order to find the step length. We have

$$\begin{aligned}\delta^-(\lambda) &= r_{24}f'_4(x_4) + c'_{24}(x_{24}) - r_{24}f'_4(x_4 - r_{24} \cdot \lambda) - c'_{24}(x_{24} - \cdot\lambda) = \\ &= 1 \cdot 0 + 0 - 1 \cdot 0 - 0 = 0,\end{aligned}$$

and

$$\begin{aligned}\delta^+(\lambda) &= r_{23}f'_3(x_3 + r_{23}\lambda) + c'_{23}(x_{23} + \lambda) - r_{23}f'_3(x_3) - c'_{23}(x_{23}) = \\ &= 0.8(15/14(x_3 + 0.8\lambda) - 10) + 0.7(x_{23} + \lambda) - 0.8(15/14x_3 - 10) - 0.7x_{23} = \\ &= 0.8 \cdot (15/14(0 + 0.8\lambda) - 10) + 0.7(0 + 0.8\lambda) - 0.8 \cdot (15/14 \cdot 0 - 10) - 0.7 \cdot 0 = \\ &= 1.3857\lambda\end{aligned}$$

x_{ij}	$j=1$	$j^{\star\star}=2$	$j^\star=3$	$j=4$	$f(x)$	102.91
$i=1$	0	0	0	10	α	5.4
$i^\star=2$	0	0	5.7732	6.2268	λ	2.1945
k_{ij}	$j=1$	$j^{\star\star}=2$	$j^\star=3$	$j=4$	v_i	w_i
$i=1$	-3.6	-4.8	-3.5361	0	-4.8	4.8
$i^\star=2$	-3.2	-5.4	0	0	-5.4	5.4

Table 2: *Numerical example - Iteration 1*

The equation (8) takes the form: $1.3857\lambda = 8$, hence $\lambda^\star = 8/1.3857 = 5.7732$. We transform the solution using the formulae (10), to obtain the values as in Table 2

$$x_{24} \leftarrow x_{24} - \lambda^\star = 12 - 5.7732 = 6.2268,$$
$$x_{23} \leftarrow x_{23} + \lambda^\star = 0 + 5.7732 = 5.7732,$$
$$x_4 \leftarrow x_4 - r_{24}\lambda^\star = 22 - 1 \cdot 5.7732 = 16.2268,$$
$$x_3 \leftarrow x_3 + r_{23}\lambda^\star = 0 + 0.8 \cdot 5.7732 = 4.6186,$$
$$k_{14} \leftarrow c'_{14}(x_{14}) + r_{14}f'_4(x_4) = 0 + 1 \cdot 0 = 0,$$
$$k_{24} \leftarrow c'_{24}(x_{24}) + r_{24}f'_4(x_4) = 0 + 1 \cdot 0 = 0,$$
$$k_{13} \leftarrow c'_{13}(x_{13}) + r_{13}f'_3(x_3) = 0.5 \cdot 0 + 0.7(15/14 \cdot 4.6186 - 10) = -3.5361,$$
$$k_{23} \leftarrow c'_{23}(x_{23}) + r_{23}f'_3(x_3) = 0.7 \cdot 5.7732 + 0.8(15/14 \cdot 4.6186 - 10) = 0.$$

The other values remain unchanged. Tables 3 and 4 present the situation after the next two iterations. The notation is as before. Observe that in Iteration 2, again $i^\star = 2$, but the slack variables are not involved this time ($j^\star = 3$ and $j^{\star\star} = 2$). In Iteration 3, the movement of the flow is made in the first row ($i^\star = 1$).

x_{ij}	$j=1$	$j=2$	$j^{\star\star}=3$	$j^\star=4$	$f(x)$	96.98
$i^\star=1$	0	0	0	10	α	4.8528
$i=2$	0	2.1945	3.5787	6.2268	λ	4.7344
k_{ij}	$j=1$	$j=2$	$j^{\star\star}=3$	$j^\star=4$	v_i	w_i
$i^\star=1$	-3.6	-3.4833	-4.8528	0	-4.8528	4.8528
$i=2$	-3.2	-3.0409	-3.0409	0	-3.2	3.2

Table 3: *Numerical example - Iteration 2*

x_{ij}	$j^{\star\star}=1$	$j=2$	$j^\star=3$	$j=4$	$f(x)$	85.49
$i^\star=1$	0	0	4.7344	5.2656	α	3.6
$i=2$	0	2.1945	3.5787	6.2268	λ	2.2086
k_{ij}	$j^{\star\star}=1$	$j=2$	$j^\star=3$	$j=4$	v_i	w_i
$i^\star=1$	-3.6	-3.4833	0	0	-3.6	3.6
$i=2$	-3.2	-3.0409	-0.2003	0	-3.2	3.2

Table 4: *Numerical example - Iteration 3*

To present this example only as briefly as possible, we switch now to the situation after Iteration 13 (Table 5). There is an interesting issue here. According to the

x_{ij}	$j^{\star\star}=1$	$j=2$	$j=3$	$j^{\star}=4$	$f(x)$	64.74
$i=1$	2.5043	3.2351	4.2606	0	α	0.2486
$i^{\star}=2$	4.8806	3.0399	3.8416	0.238	λ	0.238(0.5919)
k_{ij}	$j^{\star\star}=1$	$j=2$	$j=3$	$j^{\star}=4$	v_i	w_i
$i=1$	-3.3279	-0.2802	-0.3279	0	-0.3279	0.0478
$i^{\star}=2$	-0.2486	-0.1911	-0.1203	0	-0.2486	0.2486

Table 5: *Numerical example - Iteration 13*

x_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$f(x)$	64.7
$i=1$	2.5043	3.2351	4.2606	0	α	0.0857
$i=2$	5.1185	3.0399	3.8416	0	λ	STOP
k_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	v_i	w_i
$i=1$	-0.2422	-0.2802	-0.3279	0	-0.3279	0.0857
$i=2$	-0.1487	-0.1911	-0.1203	0	-0.1911	0.0708

Table 6: *Numerical example - Iteration 14*

formula (8), $\lambda^{\star} = 0.5919$. However, $\lambda^{\star} = 0.5919 > 0.238 = x_{24} = x_{i^{\star}j^{\star}}$, so we use the formula (9) to obtain $\lambda^{\star} = x_{i^{\star}j^{\star}} = x_{24} = 0.238$.

The results of Iteration 14 are presented in Table 6. As can be seen, the optimal solution has been reached ($\alpha = 0.0857 < 0.1 = \varepsilon$).

6. Computational Results

Some test problems were randomly generated and solved with the proposed method. Two types of cost functions f_j were considered. In the case of quadratic functions $f_j(x_j) = q_{1j}x_j^2 + q_{2j}x_j + q_{3j}$, the parameters q_{1j} were chosen randomly from the interval $[0.15, 0.66)$, the parameters q_{2j} from the interval $[-2, -1)$ and the parameters q_{3j} from the interval $[37.5, 100)$. In the case of exponential functions $f_j(x_j) = q_{1j}\exp(q_{2j}x_j) + q_{3j}x_j$, the parameters q_{1j} were chosen randomly from the interval $[10, 24)$, the parameters q_{2j} from the interval $[-0.6, -0.5)$ and the parameters q_{3j} from the interval $[1, 2)$ In both cases the transportation cost functions had the form $c_{ij}(x_{ij}) = q_{ij}x_{ij}^2$, where the parameters q_{ij} were chosen from the interval $[0.5, 1)$.The reduction ratios were chosen from the interval $[0.8, 0.9)$ and the supply from each source point from the interval $[10, 20)$. In the case of the problems attributed with quadratic cost, the optimal step length was derived using the simplified formula, while in the case of the exponential functions, the one-dimensional Newton method was used. The assumed accuracy level was set to $\varepsilon = 0.0001$. The algorithm was implemented in Java SE and run on a standard PC with an Intel(R) Core(TM) i7-2670 QM CPU @2.20 GHz. For both types of distributions, 1000 randomly generated problems of four sizes were solved: $(m, n) = (10, 10), (10, 20), (100, 100), (100, 200)$ - 8000 test problems in total. The running times in seconds (average, standard deviation, minimum and maximum) are presented in the tables below.

PROBLEM SIZE ($m \times n$)	10×10	10×20	100×100	100×200
AVG	0.00107	0.00119	2.730	16.340
ST DEV	0.00009	0.00005	0.191	1.539
MIN	0.00080	0.00110	2.011	11.405
MAX	0.00170	0.00190	2.881	19.921

Table 7: *Computing times in seconds - quadratic cost functions*

PROBLEM SIZE ($m \times n$)	10×10	10×20	100×100	100×200
AVG	0.00576	0.02624	40.700	145.551
ST DEV	0.00070	0.00536	7.443	30.725
MIN	0.00423	0.01467	28.689	117.765
MAX	0.00961	0.03558	48.033	246.872

Table 8: *Computing times in seconds - exponential cost functions*

As we can see, the algorithm is very fast - the running times are much less than a second for smaller problems and no more than few minutes for bigger ones. What is also evident is that the usage of a simplified method of calculating the step length (where possible) results in much shorter solution times.

7. Conclusion

The problem considered in this article was to find an effective algorithm for the Nonlinear Generalized Transportation Problem with convex costs. The Equalization Method presented in this paper solves the problems of this kind. It is convergent to a KKT point, i.e. to a global minimum in this case. The empirical evidence shows that the method solves quite big instances (up to $20,000$ variables) in a reasonable time.

There are two limitations of the proposed method. Firstly, it is dedicated for the problems without demand constraints. Secondly, all the cost functions are assumed to be convex and differentiable.

These two limitations allow us to define at least two possible interesting ways of developing the method. The first possible improvement would be to develop a method to solve problems with additional constraints (in particular the demand constraints on the side of destination points). The second problem possibly considered would be to adapt the method for other families of functions, for example, quasi-convex or non-differentiable functions.

References

[1] Ahuja R.K., Magnanti T.L. and Orlin J.B. (1993). Network Flows. Theory, Algorithms and Applications, Prentice Hall.

[2] Anholcer M. (2005). Zbieżność Metody wyrównań dla nieliniowych zadań alokacji (Convergence of the equalization method for the nonlinear allocation problems). In: Piasecki K., Sikora W. (eds.), Z prac Katedry Badań Operacyjnych (Research of the

Department of Operations Research), Zeszyty Naukowe Akademii Ekonomicznej w Poznaniu (Scientific Works of the Poznań University of Economics), 64, 183–198.

[3] Anholcer M. (2008). Analiza porównawcza wybranych algorytmów rozwiązywania nieliniowych zadań alokacji dóbr jednorodnych (Comparative Analysis of Chosen Algortithms for Nonlinear Problems of Allocation of Uniform Goods), Wydawnictwo Akademii Ekonomicznej w Poznaniu (Poznań University of Economics Publishing House).

[4] Anholcer M. (2008). Porównanie działania wybranych algorytmów rozwiązywania nieliniowych zadań alokacji (Comparison of the Performance of Chosen Algorithms for the Nonlinear Allocation Problems). In: Kopańska-Bródka R. (ed.), Metody i zastosowania badań operacyjnych '2007 (Methods and Applications of Operations Research '2007), Prace Naukowe Akademii Ekonomicznej w Katowicach (Scientific Works of the Katowice University of Economics), Katowice 2008, 9–25.

[5] Anholcer M. (2012). Algorithm for the stochastic generalized transportation problem. Operations Research and Decisions, 22(4), 9–20.

[6] Anholcer M. (2013). Stochastic generalized transportation problem with discrete distribution of demand. Operations Research and Decisions, 23(4), 9–19.

[7] Anholcer M., Kawa A. (2012). Optimization of supply chain via reduction of complaints ratio. Lecture Notes in Computer Science, 7327/2012, 622–628. doi:10.1007/978-3-642-30947-2_67.

[8] Balas E. (1966). The dual method for the generalized transportation problem. Management Science, 12(7), 555–568. doi:10.1287/mnsc.12.7.555.

[9] Balas E. and Ivanescu P.L. (1964). On the generalized transportation problem. Management Science, 11(1), 188–202.

[10] Bazaraa M.S., Sherali H.D. and Shetty C.M. (2006). Nonlinear Programing. Theory and Algorithms. Third Edition, Wiley-Interscience, John Wiley & Sons Inc.

[11] Cheng Y.C. (1987). Dual gradient method for linearly constrained, strongly convex, separable mathematical programming problems. Journal of Optimization Theory and Applications, 53(2), 237–246. doi:10.1007/bf00939216.

[12] Cao B. and Uebe G. (1995). Solving transportation problems with nonlinear side constraints with tabu search. Computers and Operations Research, 22(6), 593–603. doi:10.1016/0305-0548(94)00055-d.

[13] Cohen E. and Megiddo N. (1994). New algorithms for generalized network flows. Mathematical Programming, 64, 325–336. doi:10.1007/bf01582579.

[14] Cosares S. and Hochbaum D.S. (1994). Strongly polynomial algorithms for the quadratic transportation problem with a fixed number of sources. Mathematics of Operations Research, 19(1), 94–111.

[15] Dangalchev C.A. (1996). Partially-linear transportation problems. European Journal of Operational Research 91, 623–633. doi:10.1016/0377-2217(94)00367-x.

[16] Descrochers M., Marcotte P. and Stan M. (1995). The congested facility location problem. Location Science, 3(1), 9–23. doi:10.1016/0966-8349(95)00004-2.

[17] Dobre C. (2006). Convex cost flow. Adaptation of network simplex algorithm. Proceedings of ICCC 2006, Băile Felix Spa-Oradea, Romania, 1-3 June 2006.

[18] Glover F., Klingman D. and Napier A. (1972). Basic dual feasible solutions for a class of generalized networks. Operations Research, 20(1), 126–136.

[19] Goldberg A.V., Plotkin S.A. and Tardos E. (1988). Combinatorial algorithms for the generalized circulation problem. SFCS'88 Proceedings of the 29th Annual Symposium on Foundations of Computer Science, 432–443. doi:10.1109/sfcs.1988.21959.

[20] Gupta R. (1978). Solving the generalized transportation problem with constraints. Zeitschrift fr̈ angewandte Mathematik und Mechanik (Journal of Applied Mathematics

and Mechanics), 58(10), 451–458. doi:10.1002/zamm.19780581006.
[21] Harkness J. and ReVelle C. (2003). Facility location with increasing production costs. European Journal of Operational Research 145, 1–13. doi:10.1016/s0377-2217(02)00176-5.
[22] Hartwick J.M. (1970). The generalized transportation problem as a quadratic program. Queen's Economics Department Working Paper No. 35.
[23] Holmberg K. (1995). Efficient decomposition and linearization methods for the stochastic transportation problem. Computational Optimization and Applications, 4, 293–316. doi:10.1007/bf01300860.
[24] Holmberg K. and Jörnsten K. (1984). Cross decomposition applied to the stochastic transportation problem. European Journal of Operational Research, 17, 361–368. doi:10.1016/0377-2217(84)90131-0.
[25] Holmberg K., and Tuy H. (1999). A production-transportation problem with stochastic demand and concave production costs, Mathematical Programming, 85, 157–179. doi:10.1007/s101070050050.
[26] Ilich N. and Simonovic S.P. (2001). An evolution program for non-linear transportation problems. Journal of Heuristics, 7, 145–168.
[27] Jo J.-B., Li Y. and Gen M. (2007). Nonlinear fixed charge transportation problem by spanning tree-based genetic algorithm. Computers & Industrial Engineering, 53, 290–298. doi:10.1016/j.cie.2007.06.022.
[28] Kannan G., Sasi Kumar P. and Vinay V.P. (2007). Comments on "nonlinear fixed charge transportation problem by spanning tree-based genetic algorithmŤ by Jung-Bok Jo, Yinzhen Li, Mitsuo Gen, Computers & Industrial Engineering. Computers & Industrial Engineering, 55, 533–534.
[29] Klanšek U. and Pšunder M. (2010). Solving the nonlinear transportation problem by global optimization. Transport, 25(3), 314–324. doi:10.3846/transport.2010.39.
[30] Lourie J.R. (1964). Topology and computation of the generalized transportation problem. Management Science, 11(1), 177–187. doi:10.1287/mnsc.11.1.177.
[31] MacKinnon J.G. (1975). An algorithm for the generalized transportation problem. Regional Science and Urban Economics, 5(4), 445–464. doi:10.1016/0166-0462(75)90020-4.
[32] Mandayam C.V. and Prabhakar B. (2013). Road traffic networks: Optimal transport and incentives. Fifty-first Annual Allerton Conference, Allerton House, UIUC, Illinois, USA, October 2-3, 2013.
[33] Moanta D. A Three-Dimensional Nonlinear Transportation Problem, preprint.
[34] Monma C.M., Segal M. (1982). A primal algorithm for finding minimum-cost flows in capacitated networks with applications. The Bell System Technical Journal, 61(6), 949–968. doi:10.1002/j.1538-7305.1982.tb04325.x.
[35] Nguyen V.A. and Tan Y.-P. (2003). Minimum convex cost flow problem. ICICS-PCM 2003, 15-18 December 2003, Sngapore.
[36] Qi L. (1985). Forest iteration method for stochastic transportation problem. Mathematical Programming Study, 25, 142–163.
[37] Qi L. (1987). The A-forest iteration method for the Stochastic Generalized Transportation Problem. Matematics of Operations Research Vol(1), 1–21. doi:10.1287/moor.12.1.1.
[38] Patriksson M. (2008). A survey on the continuous nonlinear resource allocation problem. European Journal of Operational Research, 185, 1–46. doi:10.1016/j.ejor.2006.12.006.
[39] Romeijn E. and Sargut F.Z, (2011). The stochastic transportation problem with single sourcing. European Journal of Operational Research, 214(2), 262–272.

doi:10.1016/j.ejor.2011.04.040.

[40] Roughgarden T. (2003). The price of anarchy is independent of the network topology. Journal of Computer and System Sciences, 67(2), 341–364. doi:10.1016/s0022-0000(03)00044-8.

[41] Roughgarden T. and Tardos E. (2002). How Bad Is Selfish Routing? Journal of the ACM, 49(2), 236–259. doi:10.1145/506147.506153.

[42] Roughgarden T. and Tardos E. (2004). Bounding the inefficiency of equilibria in nonatomic congestion games. Games and Economic Behavior 47(2), 389–403. doi:10.1016/j.geb.2003.06.004.

[43] Rowse J. (1981). Solving the generalized transportation problem. Regional Science and Urban Economics, 11, 57–68.

[44] Sikora W. (1993). Problem transportowy z losowym popytem odbiorców (Transportation Problem with Stochastic Demand). Przegląd Statystyczny (Statistical Review) XXXIX (3-4), 351–364.

[45] Sikora W. (1993). Modele i metody optymalnej dystrybucji dóbr (Models and Methods of Optimal Distribution of Goods), Zeszyty naukowe - seria II, Prace doktorskie i habilitacyjne (Scientific Works Series II - Doctoral and Habilitation Dissertations), Akademia Ekonomiczna w Poznaniu (Poznań University of Economics), Poznań.

[46] Sikora W., Runka H., Pyrzyński D. (1991). Optymalizacja przepływów w sferze dystrybucji dóbr jednorodnych (Optimization of Flows in the Area of Distribution of Uniform Goods), Grant no. H\12\209\90-2, Akademia Ekonomiczna w Poznaniu (Poznań University of Economics), Poznań.

[47] Szwarc W. (1964). The transportation problem with stochastic demand. Management Science, 11(1), 33–50. doi:10.1287/mnsc.11.1.33.

[48] Tuy H., Ghannadan S., Migdalas A. and Värbrand P. (1993). Strongly polynomial algorithms for a production-transportation problem with concave production cost. Optimization 27, 205–227. doi:10.1080/02331939308843882.

[49] Tuy H., Ghannadan S., Migdalas A. and Värbrand P. (1996). A strongly polynomial algorithm for a concave production-transportation problem with a fixed number of nonlinear variables. Mathematical Programming, 72(3), 229–258. doi:10.1007/bf02592091.

[50] Walther H. (1984). Ten Applications of Graph Theory, Mathematics and Its Applications, Volume 7, Springer.

[51] Wayne K.D. (2002). A polynomial combinatorial algorithm for generalized minimum cost flow. Mathematics of Operations Research, 27(3), 445–459.

[52] Williams A.C. (1963). A stochastic transportation problem. Operations Research, 11(5), 759-770. doi:10.1287/opre.11.5.759.