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On the closed-form solution of the monopolist long-run profit maximization problem with linear demand and Cobb-Douglas technology

Zrinka Lukač[1,*], **Mira Krpan**[1] and **Vedran Kojić**[1]

[1] *Faculty of Economics and Business, University of Zagreb*
Trg J. F. Kennedyja 6, 10000 Zagreb, Croatia
E-mail: ⟨{zlukac, mkrpan, vkojic}@efzg.hr⟩

Abstract. This paper provides an economic and mathematical analysis of the long-run profit maximization problem of a monopolist with linear demand and a two-input Cobb-Douglas technology to derive the conditions that guarantee the solution to this problem. In addition, the conditions under which the closed–form solution can be derived and the conditions under which the monopolist's profit is positive are discussed. Whenever the problem has a unique solution, the closed–form defines the profit function well for the given demand function. The closed–form solution to the problem depends on the returns to scale. The problem has a unique solution for decreasing returns to scale, for which in general no closed–form can be found. On the other hand, the closed–form of the unique solution can easily be found for constant returns to scale. However, three sub-cases are identified for the case of increasing returns to scale. The analysis is supported by economic interpretations and numerical illustrations.

Keywords: closed–form solution, Cobb-Douglas technology, linear demand function, long-run profit maximization, monopoly

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1. Introduction

The monopoly maximization problem is essential component of any microeconomic analysis concerning market power, serving as a benchmark in situations where there is no competition and the monopolist controls the entire market demand . It is also a standard element of the economics of regulation, as it represents a reasonable assumption in economic environments dominated by a single large player in single-product markets, even in multi-product markets where neither demand nor costs are linked across markets. In more broader contexts (e.g. the Cournot competition model), it is analyzed as a special case [2, 4, 5, 7, 9, 15].

This paper presents the theoretical analysis of the long-run profit maximization problem faced by a monopolist using linear demand functions and the Cobb-Douglas technology with two inputs. The aim of the paper is to find a closed-form solution to this problem and to analyze the conditions under which the monopolist can achieve positive profits in the long-run. The combination of mathematical analysis and economic interpretation enabled a unique theoretical analysis of the extremely important economic problem of a monopolist's long-run profit maximization at different returns to scale. The practical implications of this work lie in defining and elaborating the conditions under which a solution to the specified long-run

*Corresponding author.

profit maximization problem of a monopolist exists. In addition, the conditions under which a closed-form solution exists, allowing a comparative static analysis are identified.

In a monopoly market, there is only one active seller. In this paper, we assume that no inactive sellers exist in the market (for the theory of contestable markets, see [3]). We also assume that the monopolist is not subject to regulation (for more on regulation theory, see [7]). Moreover, there is no price discrimination in our model, meaning the monopolist charges the same price for each unit of the good. For more information on the monopolist's price discrimination strategies, see [16]. Under these assumptions, the monopolist can choose the profit-maximizing price and quantity combination. If he chooses the profit-maximizing quantity (as in our case), the demand function determines the price charged.

Profit maximization model of a monopolist with a linear inverse demand function is often analyzed in microeconomics theory [6]. However, the monopolist's technology is usually represented by the Cobb-Douglas production function with a specified scale elasticity parameter. To the best of our knowledge, the literature lacks a comprehensive mathematical and economic analysis of the long-run profit maximization problem of a monopolist with linear demand and a two-input Cobb-Douglas production function aimed at deriving the conditions that guarantee a solution to this problem. Additionally, the conditions under which a closed–form solution can be derived and those ensuring positive profit for the monopolist's profit are discussed (here, we define a formula to be in closed–form if it can be expressed using a finite number of operations, including addition, subtraction, multiplication, division, exponentiation to a natural power, and the extraction of roots of natural degree, with the given real coefficients.) Therefore the goal of this paper is to fill this gap. In the framework presented in [17], a perfectly competitive industry uses the monopolist's product as an input. The industry's technology is represented by a CES production function, which includes the substitution parameter. The industry's derived demand function for the monopolist's good is then used in the monopolist's long-run profit maximization problem to analyze how the substitution parameter affects the monopolist's profit margin. The monopolist's technology follows the Cobb-Douglas type.

It is known from the duality theory of microeconomic analysis that the technology of a firm can be equally well represented by its production function and its cost function under certain regularity conditions [8]. Since the monopolist is assumed to be a price taker in the input market, the profit maximization model of a monopolist can be represented in two equivalent ways. In the first model, the monopolist maximizes his profit, by choosing input quantities, with technology represented by the Cobb-Douglas production function. The optimal input quantities are then substituted into the production function to obtain the optimal quantity of output that the monopolist offers to the market given the linear inverse demand function. The starting point of the second model is the derived cost function for the Cobb-Douglas production function. In the profit maximization model, the choice variable is the quantity produced. It is shown that both approaches reduce the problem to solving the equation equating marginal cost and marginal revenue, a well-known necessary condition for profit maximization.

Moreover, the economic analysis and subsequent economic interpretation provide valuable insights into the relationship between returns to scale and the monopolist's profit. Finally, the dual approach to the long-run profit maximization model of a monopolist enriches the analysis and allows us to illustrate this relationship between the monopolist's profit and returns to scale within our framework.

The structure of the paper is as follows. After the introduction and notation, the long-run profit maximization problem of a monopolist with linear demand and a two-input Cobb-Douglas production function is presented. Whenever the problem has a unique solution, the closed–form defines the profit function for the given demand function. It is shown that the closed–form solution to the problem is conditional on economies of scale when demand is linear. This generalization of the monopolist's long-run profit maximization model with linear demand and Cobb-Douglas technology provides a thorough economic and mathematical analysis enriched by economic interpretations. Conditions under which the analyzed problem has a closed–form solution are identified, along with the market environment in which the monopolist continues producing in the long-run. Before the conclusions, some numerical examples are provided.

2. Model and main results

The model assumes that the monopolist faces a downward-sloping linear market demand function for the good it produces

$$p(q) = a - bq, \tag{1}$$

where $a, b > 0$, and p is the product price at which the monopolist can sell q units of the product he produces. The technology of a monopolist is represented by the Cobb-Douglas production function:

$$q = q(x_1, x_2) = x_1^{\alpha} x_2^{\beta}, \tag{2}$$

where q is the quantity of production, $x_1, x_2 \geq 0$ are input quantities, $\alpha > 0$ is the output elasticity of the first input, and $\beta > 0$ is the output elasticity of the second input.

It is assumed that the monopolist is the price taker in the input market, so the input prices $w_1, w_2 > 0$, are parameters of the model and the monopolist takes them as given.

The value of the elasticity of scale, ε, which is a measure of returns to scale for Cobb-Douglas technology, is $\varepsilon = \alpha + \beta$ [12]. Eatwell [10] defines returns to scale as follows: *"The production technique of a good y can be characterized as a function of the inputs x required: If all inputs are multiplied by a positive scalar t and the resulting output is represented as $t^s y$, then the value of s can be taken as an indicator of the level of returns to scale. If $s = 1$, then there are constant returns to scale: Any proportional change in all inputs results in an equally proportional change in output."* Elasticity of scale is greater than 1 when returns to scale are increasing, and less than 1 when returns to scale are decreasing [12].

The monopolist thus chooses the profit-maximizing quantity of production factors and the production quantity. In this context, the profit maximization model can be written in two ways, depending on whether we first want to derive the monopolist's input demand functions for a given demand function or determine the optimal production quantity directly. This distinction arises from duality results in production theory, where the producer's technology can be equivalently represented by either the production function or the cost function under certain regularity conditions. Both formulations are given below.

The long-run profit maximization model of a monopolist when the quantities of the factors of production are the choice variables is represented as:

$$\max_{x_1, x_2 \geq 0} p(q(x_1, x_2)) \cdot q(x_1, x_2) - w_1 x_1 - w_2 x_2, \tag{3}$$

where $w_1, w_2 > 0$ are input prices. In this approach, the technology of a monopolist is represented by the production function. Alternatively, the long-run profit maximization model of a monopolist can be represented in the following way:

$$\max_{q > 0} \Pi(q) = p(q)q - c(q), \tag{4}$$

where $c(q)$ is the cost function. It is known from microeconomic theory that the cost function is the result of an optimization problem in which the producer minimizes costs while taking into account a given level of production.

We demonstrate that both approaches reduce the discussion to the solvability of the equation equating marginal cost and marginal revenue, a well-known necessary condition for profit maximization. Additionally, the following result will be used in the remainder of the paper.

Theorem 1. *(Abel's Theorem) The generic algebraic equation of degree higher than four is not solvable by radicals, i.e., formulae do not exist for expressing roots of a generic equation of degree higher than four in terms of its coefficients by means of operations of addition, subtraction, multiplication, division, raising to a natural power, and extraction of a root of natural degree.*

The proof of Abel's theorem can be found in [1].

2.1. The first approach to solving the long-run profit maximization model of a monopolist

Below the first-order necessary conditions for the model (3) are derived:

$$\frac{\partial p}{\partial x_1}q + p\frac{\partial q}{\partial x_1} - w_1 = 0, \tag{5}$$

$$-b\frac{\partial q}{\partial x_1}q + p\frac{\partial q}{\partial x_1} - w_1 = 0, \tag{6}$$

and

$$\frac{\partial p}{\partial x_2}q + p\frac{\partial q}{\partial x_2} - w_2 = 0, \tag{7}$$

$$-b\frac{\partial q}{\partial x_2}q + p\frac{\partial q}{\partial x_2} - w_2 = 0. \tag{8}$$

Therefore,

$$\frac{\partial q}{\partial x_1}(p - bq) = w_1, \tag{9}$$

$$\frac{\partial q}{\partial x_2}(p - bq) = w_2. \tag{10}$$

Differentiation of the production function with respect to the quantities of inputs gives the marginal products of inputs:

$$\frac{\partial q}{\partial x_1} = \alpha \cdot \frac{q}{x_1}, \tag{11}$$

$$\frac{\partial q}{\partial x_2} = \beta \cdot \frac{q}{x_2}. \tag{12}$$

From (9) and (10) the equality between the input price ratio on the left-hand side and the marginal rate of substitution on the right-hand side is derived

$$\frac{w_1}{w_2} = \frac{\frac{\partial q}{\partial x_1}}{\frac{\partial q}{\partial x_2}} = \frac{\alpha\frac{q}{x_1}}{\beta\frac{q}{x_2}} = \frac{\alpha}{\beta} \cdot \frac{x_2}{x_1}. \tag{13}$$

From (13) the long-run production expansion path follows:

$$x_2 = \frac{\beta}{\alpha} \cdot \frac{w_1}{w_2}x_1. \tag{14}$$

Inserting (11) into (9) gives

$$x_1 = \frac{\alpha}{w_1} \cdot q(p - bq). \tag{15}$$

Due to the symmetry of the initial problem in (3), we get by analogy

$$x_2 = \frac{\beta}{w_2} \cdot q(p - bq). \tag{16}$$

From (1) the marginal revenue function is obtained:

$$MR(q) = \frac{d(p(q)q)}{dq} = \frac{d(aq - bq^2)}{dq} = a - 2bq = (a - bq) - bq = p - bq. \tag{17}$$

If (17) is inserted into (15) and (16), (15) and (16) become

$$x_1 = \frac{\alpha}{w_1} \cdot q(a - 2bq), \tag{18}$$

$$x_2 = \frac{\beta}{w_2} \cdot q(a - 2bq). \tag{19}$$

For the most interesting case where $x_1 > 0$ and $x_2 > 0$, from (18) and (19) it can be concluded that the following inequality has to be satisfied:

$$a - 2bq > 0 \Longleftrightarrow q \in \left\langle 0, \frac{a}{2b} \right\rangle. \tag{20}$$

Inserting (18) and (19) in (2) gives

$$q = \left(\frac{\alpha}{w_1}\right)^{\alpha} \left(\frac{\beta}{w_2}\right)^{\beta} q^{\alpha+\beta} (a - 2bq)^{\alpha+\beta}. \tag{21}$$

From (21) the following equation is obtained

$$a - 2bq = \rho q^{\frac{1}{\alpha+\beta} - 1}, \tag{22}$$

where the right hand side of equation (22), as we will see below, is the marginal cost of production function for Cobb-Douglas technology, $MC(q)$, and ρ is the marginal cost of 1 unit of output,

$$\rho = \left[\left(\frac{w_1}{\alpha}\right)^{\alpha} \left(\frac{w_2}{\beta}\right)^{\beta} \right]^{\frac{1}{\alpha+\beta}}. \tag{23}$$

Therefore, the term in (22) expresses the equality between marginal revenue (on the left) and marginal cost (on the right), $MR(q) = MC(q)$, the very well known first-order necessary condition for profit maximization. In the next subsection, the same equation is derived using a different approach, and the rest of the paper discusses its solvability.

2.2. The second approach to solving the long-run profit maximization model of a monopolist

In this second approach, the technology is represented by the cost function instead of the production function [8]. In the first step, the cost function is derived and then, in the second step, it is inserted into the long-term profit maximization model with the production quantity as the choice variable. In the following, the cost minimization model is solved to obtain the cost function for the Cobb-Douglas technology.

The cost function is derived from the cost minimization model subject to the given level of production [12]:

$$\begin{cases} \min\limits_{x_1, x_2 \geq 0} & w_1 x_1 + w_2 x_2 \\ \text{s.t.} & q(x_1, x_2) = x_1^{\alpha} x_2^{\beta} = q\,. \end{cases} \tag{24}$$

The solution of the model is the set of conditional input demand functions, $x_1(w_1, w_2, q)$ and $x_2(w_1, w_2, q)$, and the value function of the model is the cost function, $c(w_1, w_2, q) = w_1 x_1(w_1, w_2, q) + w_2 x_2(w_1, w_2, q)$.

The cost function for Cobb-Douglas technology is derived analytically in [17], from which we get the functional forms for the conditional demand functions for inputs and the cost function as given below:

$$x_1(w_1, w_2, q) = \left(\frac{\alpha}{\beta} \cdot \frac{w_2}{w_1}\right)^{\frac{\beta}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}}, \tag{25}$$

$$x_2(w_1, w_2, q) = \left(\frac{\beta}{\alpha} \cdot \frac{w_1}{w_2}\right)^{\frac{\alpha}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}}, \tag{26}$$

and

$$c(w_1, w_2, q) = (\alpha + \beta)\left[\left(\frac{w_1}{\alpha}\right)^{\alpha}\left(\frac{w_2}{\beta}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}}. \tag{27}$$

Taking into account the linear inverse demand function and the derived cost function, the profit maximization model (3) reduces to

$$\max_{q \geq 0} = (a - bq)q - (\alpha + \beta)\left[\left(\frac{w_1}{\alpha}\right)^{\alpha}\left(\frac{w_2}{\beta}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}}, \tag{28}$$

which can also be writen as

$$\max_{q \geq 0} = aq - bq^2 - c(w_1, w_2, 1)q^{\frac{1}{\alpha+\beta}}, \tag{29}$$

where

$$c(w_1, w_2, 1) = (\alpha + \beta)\left[\left(\frac{w_1}{\alpha}\right)^{\alpha}\left(\frac{w_2}{\beta}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} > 0 \tag{30}$$

is the unit cost function, or the cost of 1 unit of output [12]. Given that $p(q) \geq 0$, $a - bq \geq 0$, it follows that $q \leq \frac{a}{b}$. Next, we derive the necessary first-order conditions by differentiating the objective function in terms of q and setting this derivative equal to zero,

$$\Pi'(q) = a - 2bq - c(w_1, w_2, 1)\frac{1}{\alpha + \beta} q^{\frac{1}{\alpha+\beta}-1} = 0. \tag{31}$$

Thus,

$$a - 2bq = c(w_1, w_2, 1)\frac{1}{\alpha + \beta} q^{\frac{1}{\alpha+\beta}-1} = 0. \tag{32}$$

If we recall the definition of ρ,

$$\rho = c(w_1, w_2, 1)\frac{1}{\alpha + \beta} \text{ and } \varepsilon = \alpha + \beta > 0, \tag{33}$$

the equation (32) can be written in the same form as in (22),

$$a - 2bq = \rho q^{\frac{1}{\varepsilon}-1}. \tag{34}$$

The left-hand side is the marginal revenue function, $MR(q)$, and the right-hand side is the marginal cost function, $MC(q)$. In general, the sufficient conditions for profit maximization of the monopolist reduce to the condition that the $\frac{dMR(q)}{dq} < \frac{dMC(q)}{dq}$, or intuitively that the slope of the marginal revenue curve at the stationary point is less than the slope of the marginal cost curve.

Due to the non-negativity of marginal costs, $\rho q^{\frac{1}{\varepsilon}-1} \geq 0$, marginal revenue cannot be negative either, which brings us back to the condition given by (20). This condition implies that the monopolist cannot maximize his profit on the inelastic part of the demand curve.

The closed–form solution for q depends on the explicit solvability of the equation

$$a - 2bq - \rho q^{\frac{1}{\varepsilon}-1} = 0, \tag{35}$$

which follows from (34). Bellow we comment the solvability of equation (35) as a function of the value of elasticity of scale, as a measure of returns to scale [12].

The exponent $\frac{1}{\varepsilon} - 1$ in equation (35) is a real number. However, since the set of rational numbers $\mathbb{Q}$ is dense in the set of real numbers $\mathbb{R}$, any real number can be approximated to any given degree of accuracy by a rational number. For practical and applicative reasons, we assume that $\varepsilon > 0$ from equation (33) can be written as a positive fraction, i.e.,

$$\varepsilon = \frac{m}{n},\ \gcd(m,n) = 1,\ m,n \in \mathbb{N}. \tag{36}$$

Accordingly, equation (35) becomes

$$\rho q^{\frac{n}{m}-1} = a - 2bq. \tag{37}$$

Theorem 2. *Equation (37) has a closed-form solution if and only if*

$$\varepsilon = \frac{m}{n} \in \left\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1, \frac{3}{2}, 2\right\}. \tag{38}$$

Proof. Using the substitution $t = q^{\frac{1}{m}} > 0$, equation (37) becomes equivalent to

$$\rho t^{n-m} = a - 2bt^m. \tag{39}$$

Let us consider two cases. In the first case, if $n - m \geq 0$, then according to Abel's theorem (Theorem 1), equation (39) has a closed-form solution if and only if its degree is less than or equal to 4. Therefore, the following inequalities must hold:

$$\begin{cases} 1 \leq m \leq 4 \\ 0 \leq n - m \leq 4. \end{cases} \tag{40}$$

Since m and n are positive integers, it is straightforward to find all solutions of (40):

$$\begin{cases} m = 1 \Rightarrow n \in \{1,2,3,4,5\}, \\ m = 2 \Rightarrow n \in \{2,3,4,5,6\}, \\ m = 3 \Rightarrow n \in \{3,4,5,6,7\}, \\ m = 4 \Rightarrow n \in \{4,5,6,7,8\}. \end{cases} \tag{41}$$

Thus, in the first case, we have

$$\frac{m}{n} \in \left\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\right\}. \tag{42}$$

In the second case, where $n - m < 0$, we have $m - n > 0$. Multiplying equation (39) by t^{m-n}, it becomes equivalent to

$$\rho = at^{m-n} - 2bt^{2m-n}. \tag{43}$$

Note that $2m - n > m - n > 0$. Similarly to the first case, equation (43) has a closed-form solution if and only if its degree is less than or equal to 4. Thus, the following inequalities must hold:

$$\begin{cases} 1 \leq 2m - n \leq 4 \\ 1 \leq m - n \leq 4. \end{cases} \tag{44}$$

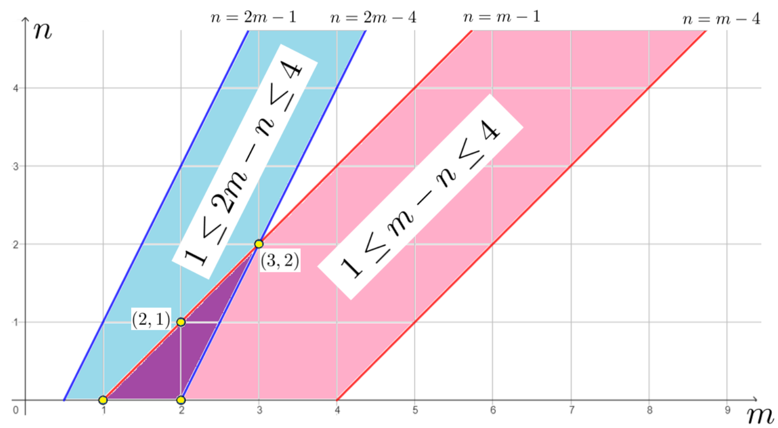

Figure 1: *Solution of the system (44).*

Using a graphical method (see Figure 1), it is easy to find the solutions of the system (44):

$$(m, n) \in \{(2, 1), (3, 2)\} \Longleftrightarrow \frac{m}{n} \in \left\{\frac{3}{2}, 2\right\}. \tag{45}$$

The union of the sets in (42) and (45) yields (38), which completes the proof. □

The closed-form solution of equation (37) allows for the derivation of explicit formulas for the input demand functions and the profit function, which are crucial for sensitivity analysis and comparative statics in microeconomics. In the remainder of the paper, we will present several numerical examples to illustrate the solvability of model (4) and equation (37) depending on the value of ε. A full illustration of all values of the parameter ε from (38) is beyond the scope of this paper.

CASE I. ($\varepsilon = 1$ – **CONSTANT RETURNS TO SCALE**)

In the case of constant returns to scale, the elasticity of scale is equal to 1, $\varepsilon = \alpha + \beta = 1$, and equation (35) becomes

$$\Pi'(q) = a - 2bq - \rho = 0, \tag{46}$$

from which it follows that

$$q = \frac{a - \rho}{2b}, \tag{47}$$

where the condition $a > \rho$ must hold. The second order necessary condition reduces to

$$\Pi''(q) = -2b < 0. \tag{48}$$

Since marginal cost is constant in the case of global constant returns to scale, its derivative is intuitively zero. The slope of the marginal revenue function is negative.

Therefore, the supply function is derived for the given demand function:

$$q = q(a, b, w_1, w_2) = \frac{a - \rho}{2b}. \tag{49}$$

The following input demand functions and the profit functions for the given demand function are obtained:

$$x_1 = x_1(a, b, w_1, w_2) = \frac{\rho\alpha}{2bw_1}(a - \rho), \tag{50}$$

$$x_2 = x_2(a, b, w_1, w_2) = \frac{\rho\beta}{2bw_2}(a - \rho), \tag{51}$$

$$\begin{aligned}\pi(a, b, w_1, w_2) &= pq - w_1x_1 - w_2x_2 \overset{(3)}{=} (a - bq)q - w_1\frac{\rho\alpha(a-\rho)}{2bw_1} - w_2\frac{\rho\beta(a-\rho)}{2bw_2}\\ &= (a - bq)q - \underbrace{(\alpha+\beta)}_{=\varepsilon=1}\rho\underbrace{\left(\frac{a-\rho}{2b}\right)}_{=q} = (a - bq)q - \rho q = (a - \rho)q - bq^2 =\\ &= (a-\rho)\frac{a-\rho}{2b} - b\Big(\frac{a-\rho}{2b}\Big)^2 = \frac{(a-\rho)^2}{4b}.\end{aligned} \tag{52}$$

In the second approach, the input demand functions for the given demand function are derived by substituting the optimal output quantity (49) into the conditional input demand functions for the cost-minimising monopolist, as given by (25) and (26). In this way, the closed–form solutions for the input demand functions, the supply function and the profit function for the given demand function are obtained. Moreover, the economic profit in (52) is positive. Intuitively, in the case of constant returns to scale, both the average and marginal costs are constant and equal. In equilibrium, the marginal cost is equal to the marginal revenue and the price exceeds marginal revenue when the demand function is decreasing. As a result, the price is higher than the average cost, ensuring the profit is positive.

CASE II. ($\varepsilon < 1$ – **DECREASING RETURNS TO SCALE**)

In the case of decreasing returns to scale the elasticity of scale is less than 1, $\varepsilon < 1$. The function $q \mapsto \rho q^{\frac{1}{\varepsilon}-1}$, which represents the marginal cost function, is strictly increasing (since $\frac{1}{\varepsilon} - 1 > 0$), but the function $q \mapsto a - 2bq$, which represents the marginal revenue function, is strictly decreasing, so that their graphs intersect only once (Figure 2). Consequently, equation (35) has a unique solution and problem (4) has a unique stationary point.

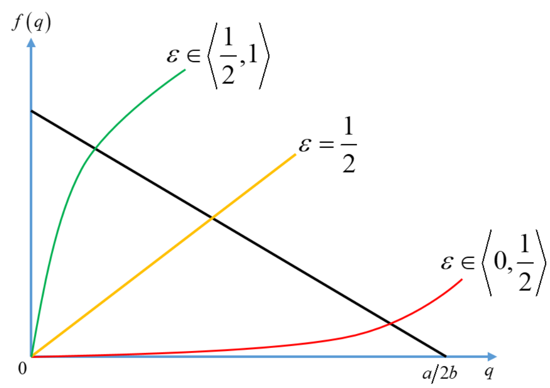

Figure 2: *The marginal revenue and the marginal cost functions in the case of decreasing returns to scale.*

For other values of ε, equation (35) can be solved by numerical methods. Moreover, the sufficient condition is always satisfied when $\alpha + \beta < 1$, since

$$\Pi''(q) = -2b - \rho\Big(\frac{1}{\varepsilon} - 1\Big)q^{\frac{1}{\varepsilon}-2} < 0. \tag{53}$$

The sufficient conditions for the local maximum are satisfied, and the unique stationary point of the problem (4) is the global maximum of the problem (4). Moreover, it can be shown that the global maximum is positive:

$$\begin{aligned} pq^* - w_1x_1^* - w_2x_2^* &= (a - bq^*)q^* - (\alpha + \beta)q^*(a - 2bq^*) \\ &= (1 - \alpha - \beta)(a - 2bq^*)q^* + b(q^*)^2 > 0. \end{aligned} \tag{54}$$

Intuitively, under globally decreasing returns to scale for a single product and perfect competition in markets for factors of production, both marginal and average costs increase as output rises, but with costs exceeding average costs. Since marginal cost equals marginal revenue in equilibrium, and price exceeds marginal revenue in a monopoly, profit must be positive in equilibrium because the price is higher than average cost.

Example 1. *We start with the value of the elasticity of scale equal to 0.5, i.e.* $\varepsilon = \frac{1}{2}$*. In this case* $\rho = \left(\frac{\alpha}{w_1}\right)^{-2\alpha}\left(\frac{\beta}{w_2}\right)^{-2\beta}$*.The equation (35) becomes* $\rho q = a - 2bq$*, and the supply function for the given demand function is derived,*

$$q = q(a, b, w_1, w_2) = \frac{a}{\rho + 2b}. \tag{55}$$

It is easy to see that $\frac{a}{\rho+2b} \in \langle 0, \frac{a}{2b}\rangle$*. The following input demand functions and profit functions for the given demand function are obtained:*

$$x_1 = x_1(a, b, w_1, w_2) = \frac{\alpha\rho}{w_1}\Big(\frac{a}{\rho + 2b}\Big)^2, \tag{56}$$

$$x_2 = x_2(a, b, w_1, w_2) = \frac{\beta\rho}{w_2}\Big(\frac{a}{\rho + 2b}\Big)^2, \tag{57}$$

$$\pi(a, b, w_1, w_2) = \Big(1 - b - \frac{\rho}{2}\Big)\Big(\frac{a}{\rho + 2b}\Big)^2. \tag{58}$$

Example 2. *Let* $\varepsilon = \frac{1}{3}$*. In this case* $\rho = \left(\frac{\alpha}{w_1}\right)^{-3\alpha}\left(\frac{\beta}{w_2}\right)^{-3\beta}$*. The equation (34) becomes* $\rho q^2 = a - 2bq$*, from which, due to* $q > 0$*, the supply function for the given demand function is obtained*

$$q = q(a, b, w_1, w_2) = \frac{1}{\rho}(\sqrt{b^2 + a\rho} - b). \tag{59}$$

It can be shown that $0 < \frac{1}{\rho}(\sqrt{b^2 + a\rho} - b) < \frac{a}{2b}$*. The following input demand functions and the profit functions for the given demand function are obtained:*

$$x_1 = x_1(a, bw_1, w_2) = \frac{\alpha}{\rho^2 w_1}\big(\sqrt{b^2 + a\rho} - b\big)^3, \tag{60}$$

$$x_2 = x_2(a, bw_1, w_2) = \frac{\beta}{\rho^2 w_2}\big(\sqrt{b^2 + a\rho} - b\big)^3, \tag{61}$$

$$\pi(a, b, w_1, w_2) = \frac{\sqrt{b^2 + a\rho} - b}{\rho}\Big(a - \frac{b}{\rho}\big(\sqrt{b^2 + a\rho} - b\big) - \frac{1}{3\rho}\big(\sqrt{b^2 + a\rho} - b\big)^2\Big). \tag{62}$$

CASE III. ($\varepsilon > 1$ – **INCREASING RETURNS TO SCALE**)

In the case of increasing returns to scale, where the scale elasticity is greater than 1, $\varepsilon = \alpha + \beta > 1$, the solvability of equation (35) is discussed below, which reducing the problem to finding the roots of the equation

$$\varphi(q) = \rho q^{\frac{1}{\varepsilon} - 1} + 2bq - a = 0. \tag{63}$$

Note that

$$\varphi(q) = -\Pi'(q). \tag{64}$$

It is easy to show that the function $q \mapsto \rho q^{\frac{1}{3}-1}$, which is the marginal cost function, is strictly decreasing (since $\frac{1}{\varepsilon} < 0$), strictly concave (due to $(\frac{1}{\varepsilon}-1)(\frac{1}{\varepsilon}-2) > 0$), as well as that the following holds: $\lim_{q\to 0+} \rho q^{\frac{1}{\varepsilon}-1} = +\infty$ and $\lim_{q\to +\infty} \rho q^{\frac{1}{\varepsilon}-1} = 0$. Moreover, the function $q \mapsto a - 2bq$, or the marginal revenue function, is strictly decreasing and linear.

Therefore, in search of the roots of the function $\varphi(q)$ three cases may occur: (a) the function φ has no roots ($\varphi > 0$ on the whole domain, Figure 3a, (b) the function φ has one root, Figure 3b, and (c) the function φ has two roots (see Figure 3c).

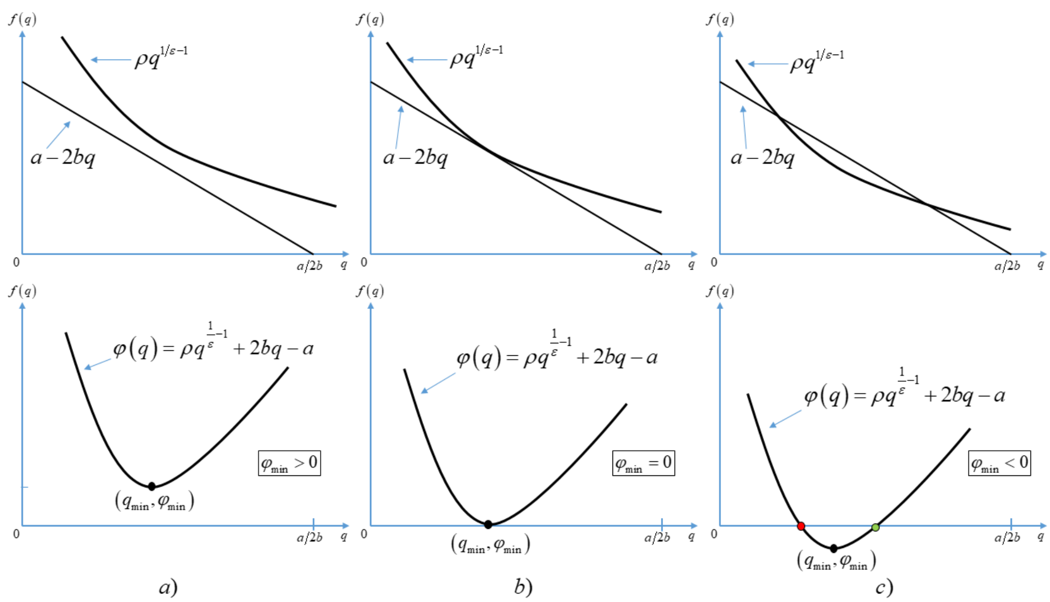

Figure 3: *The marginal revenue and the marginal cost functions in the case of increasing returns to scale.*

Subcase III.a) (the increasing returns to scale and $\varphi_{\min} > 0$)

Additionally, since $\varphi(q) = -\Pi'(q) > 0$, it follows that $\Pi'(q) < 0$. In the long-run, all costs are variable, $\Pi(0) = 0$, and the profit is negative for all $q \in \langle 0, \frac{a}{2b}\rangle$. For every unit of output, the monopolist would lose money since the marginal cost is higher than the marginal revenue.

Example 3. *Let us assume $a = b = \alpha = \beta = w_1 = w_2 = 1$. Then $\varepsilon = 2$, $\rho = 1$, and the problem (3) reduces to*

$$\max_{x_1,x_2\geq 0} p(q)\cdot q - 1\cdot x_1 - 1\cdot x_2 = (1-x_1x_2)x_1x_2 - x_1 - x_2. \tag{65}$$

The objective function (65) has no stationary points and it is negative on the whole domain (see Figure 4).

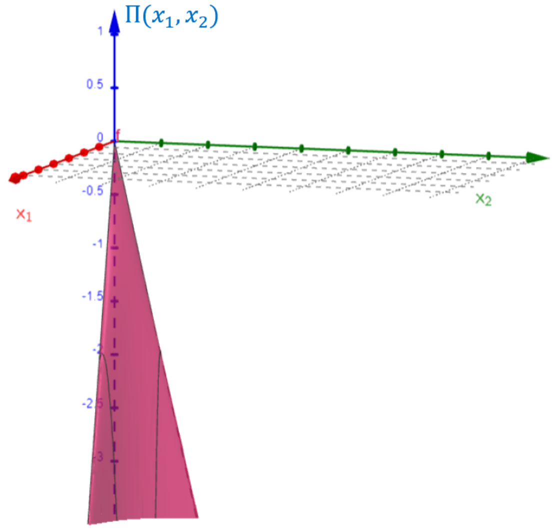

Figure 4: *Negative profit in the case of increasing returns to scale with the scale elasticity equal to 2.*

Subcase III.b) (the increasing returns to scale and $\varphi_{\min} = 0$)
In this case, $\varphi(q) = 0$. From the previous analysis, it is clear that $\varphi_{\min} = 0$ is the value of the strict global minimum of the function φ that is reached at the point $q_{\min}$. Therefore, $q_{\min}$ is both the null point of the function φ and the point of its strict global minimum. The value of $q_{\min}$ can be calculated. Namely, from $\varphi'(q_{\min}) = 0$ the following equation follows: $\left(\frac{1}{\varepsilon} - 1\right)\rho q^{\frac{1}{\varepsilon}-2} + 2b = 0$. Therefore,

$$q_{\min} = \left(\frac{2b}{\rho\left(1 - \frac{1}{\varepsilon}\right)}\right)^{\frac{\varepsilon}{1-2\varepsilon}}, \tag{66}$$

and

$$\varphi_{\min} = (2b)^{\frac{1-\varepsilon}{1-2\varepsilon}} \left(\frac{\varepsilon}{\rho(\varepsilon - 1)}\right)^{\frac{\varepsilon}{1-2\varepsilon}} \cdot \frac{2\varepsilon - 1}{\rho(\varepsilon - 1)} - a. \tag{67}$$

At this point, however, maximum profit is not obtained. Namely, from $\varphi(q_{\min} = 0)$ it follows that $\Pi'(q_{\min}) = 0$. Moreover, from $\varphi'(q_{\min}) = 0$ we have $\Pi''(q_{\min}) = 0$, and from $\varphi''(q_{\min}) > 0$ it follows that $\Pi'''(q_{\min}) \neq 0$. We can conclude that $q_{\min}$ is the inflection point.

Subcase III.c) (the increasing returns to scale and $\varphi_{\min} < 0$)
In this case,

$$\varphi_{\min} < 0 \Leftrightarrow a > (2b)^{\frac{1-\varepsilon}{1-2\varepsilon}} \left(\frac{\varepsilon}{\rho(\varepsilon - 1)}\right)^{\frac{\varepsilon}{1-2\varepsilon}} \cdot \frac{2\varepsilon - 1}{\rho(\varepsilon - 1)}. \tag{68}$$

The global minimum of the function φ is less than zero (negative), so the function φ has two null points, q_1 and q_2, where $q_1 < q_{\min} < q_2$ (Figure 3c). However, at q_1 maximim profit is not achieved. Naimly, from $\varphi(q_1) = 0$ it follows that $\Pi'(q_1) = 0$. It is clear that $\Pi'(q_1) < 0$ since φ is decreasing on $\langle 0, q_{\min}\rangle$, and $\varphi'(q_1) = -\Pi''(q_1) < 0 \Rightarrow \Pi''(q_1) > 0$. Therefore, in q_1 the function achieves local minimum profit.

On the other hand, it is clear that for q_2 it holds that $\varphi'(q_2) > 0$, so $\varphi'(q_2) = -\Pi''(q_2) > 0$ and $\Pi''(q_2) < 0$. Therefore, the function reaches local maximum profit in q_2.

Under conditions of increasing returns to scale, case c, (as specified in condition (68)), represents a necessary condition for maximizing profit. However, this condition applies to both minimum and maximum and says nothing about the positivity of the maximum profit.Consequently, this condition does not guarantee a positive profit. Furthermore, a closed–form solution for the profit function cannot be established. This result is illustrated by the following two examples.

Example 4. *Let us assume that* $a = 5, b = \alpha = \beta = w_1 = w_2 = 1$. *So* $\varepsilon = 2, \rho = 1$, *and the problem (3) becomes*

$$\max_{x_1,x_2\geq 0} p(q)q - 1\cdot x_1 - 1\cdot x_2 = (5 - x_1x_2)x_1x_2 - x_1 - x_2. \tag{69}$$

Condition (68) is satisfied. The solution to the problem (69) is the global maximum. Its value is approximately 3.19501, and it is achieved at point $(x_1, x_2) \approx (1.46962, 1.46962)$ *(Figure 5).*

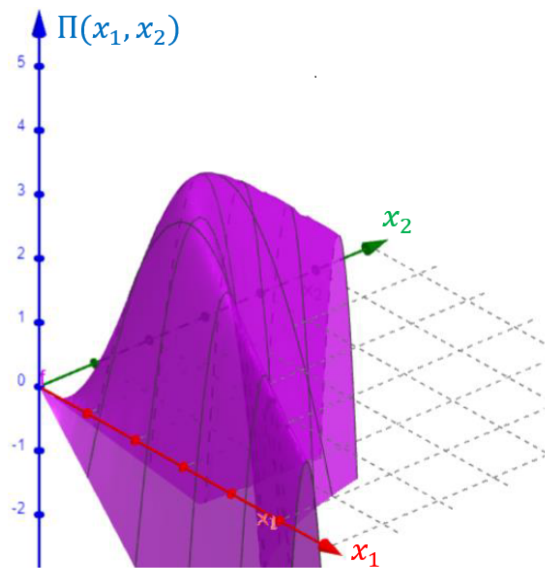

Figure 5: *The graph of the function (69).*

Example 5. *Let us assume that* $a = 5, b = 1, \alpha = \beta = w_1 = w_2 = 10$. *So* $\varepsilon = 20, \rho = 1$, *and the problem (3) becomes*

$$\max_{x_1,x_2\geq 0} p(q)q - 10\cdot x_1 - 10\cdot x_2 = (5 - x_1^{10}x_2^{10})x_1^{10}x_2^{10} - 10x_1 - 10x_2. \tag{70}$$

Condition (68) is satisfied. The solution to the problem (70) is the local maximum. Its value is approximately -14.6397, and it is achieved at point $(x_1, x_2) \approx (1.04185, 1.04185)$. *It can be concluded that the objective function of (68) is negative on the whole area (the exception is its value 0) and under these conditions the monopolist would operate at a loss.*

3. Conclusion

This paper examines the long-run profit maximization of a monopolist operating under linear demand and a two-input Cobb-Douglas production function. When the problem has a unique solution, the closed–form expression defines the profit function for the given demand function. It is shown that the closed–form solution to the problem is conditional on economies of scale in a linear demand setting. This generalization of the monopolist's long-run profit maximization model with linear demand and Cobb-Douglas technology provides a comprehensive economic and mathematical analysis enriched by economic interpretations. The conditions under which the problem yields a closed-form solution are identified, along with the market environment that enables the monopolist to sustain production in the long-run. Due to implications of duality results in production theory under certain regularity conditions, the monopolist's technology in the model is represented in two equivalent ways, by the Cobb-Douglas production function and the corresponding derived cost function. It is shown that both approaches reduce the discussion to the solvability of the equation equating marginal cost and marginal revenue, the well-known necessary condition for profit maximization. Furthermore, the economic analysis and subsequent economic interpretation offer interesting insights into the relationship between returns to scale and the monopolist's profit. The combination of mathematical analysis and economic interpretation enabled a unique theoretical analysis of the extremely important economic problem of the monopolist's long-run profit maximization under varying returns to scale. The practical implications of this work lie in defining and elaborating the conditions under which a solution to the specified long-run profit maximization problem of a monopolist exists. In addition, conditions are identified under which a closed-form solution is obtainable, enabling a comparative static analysis. Future research could focus on conducting sensitivity analysis and further comparative statics.

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