Inverse cost and revenue efficiency in network processes with uncontrollable inputs

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Abstract. Inverse data envelopment analysis (IDEA) estimates the inputs/outputs of each decision-making unit (DMU) based on the perturbations in the outputs/inputs while maintaining relative efficiency. When the cost of inputs or the price of outputs is available, it is possible to calculate cost, revenue, and profit efficiencies. This study develops an inverse network DEA that includes uncontrollable measures. For this purpose, models are presented for calculating relative, cost, and revenue efficiencies. Then, an algorithm is proposed to estimate the inputs of the first stage, considering unchanged technical and cost efficiencies. Also, an algorithm is presented to estimate the outputs of the second stage, considering unchanged output-oriented technical efficiency and revenue efficiency. Then, the introduced algorithms are applied to a numerical example and a dataset related to salmon farming, obtaining logical results. Finally, the proposed method is compared with one of the existing methods, and their differences are discussed.

Keywords: Cost efficiency, Inverse data envelopment analysis, revenue efficiency, uncontrollable inputs

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1. Introduction

Data envelopment analysis (DEA), first presented by Charnes et al. [5], is a technique for the relative efficiency calculation of decision-making units (DMUs). However, the inverse DEA (IDEA) is an approach to estimate the performance indicators (inputs and outputs) of individual units by perturbing other measures (outputs and inputs) while maintaining the unit's efficiency [21]. Yan et al. [23] further explored IDEA models by incorporating conic constraints. Subsequently, Jahanshahloo et al. [13] extended the technique provided by Yan et al. [23], utilizing IDEA to assess the output levels of a DMU under increased inputs or improved efficiency. Hadi-Vencheh and Foroughi [11] rendered a technique that involved simultaneous increases in certain inputs (outputs) while decreasing others.

Researchers can analyze the performance of individual units in terms of cost, revenue, or profit when input and output prices are known. The main models for assessing cost efficiency (CE) and revenue efficiency (RE) focus on minimizing costs and maximizing revenue. Additional models have been introduced to calculate maximum profit and determine profit efficiency. Fare et al. [7] introduced a linear programming model to estimate the cost, revenue, and profit efficiencies of DMUs. Tone [20] proposed an alternative approach for the cost, revenue, and profit efficiency evaluation considering different prices for inputs and outputs. Sahoo et al.

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[17] suggested a directional distance function method to evaluate cost, revenue, and profit efficiencies. Ghiyasi [10] developed an IDEA model for situations where price information was available, proposing models to estimate inputs and outputs while maintaining technical and cost (or revenue) efficiencies. Asadi et al. [2] introduced an IDEA approach based on non-convex cost efficiency. Soleimani Chamkhorami et al. [19] determined the minimum required changes in inputs for perturbing outputs so that the cost efficiency of the evaluated unit remained unchanged. In their research, Fathi and Izadikhah [9] and Khoshfetrat and Ghiyasi [15] examined the issue of resource allocation when non-discretionary inputs were present using radial IDEA models. Jahani Sayyad Noveiri et al. [12] introduced non-radial DEA-based models to evaluate the efficiency and output variations of Iranian restaurants, incorporating non-discretionary measures.

Neglecting the internal structure of processes and treating systems as black boxes can lead to inaccurate performance evaluations. Internal processes can significantly affect performance, which is why researchers such as Fare and Grosskopf [6] introduced the concept of network structure for DMUs in DEA. Subsequent methods, such as those developed by Kao and Hwang [14] and Wu et al. [22], have sought to assess the efficiency of two-stage networks. Recent studies, including Lozano's approach [16], have focused on analyzing the cost efficiency of two-stage processes considering intermediate factors. Banihashemi and Tohidi [4] used a slacks-based model to determine the cost, revenue, and profit efficiencies of network structures based on input and output price vectors, also extending their analysis to supply chain networks [3]. Shiri et al. [18] applied the IDEA concept to calculate cost and revenue efficiencies in two-stage structures while expanding Ghiyasi's method for two-stage networks. Despite the efforts made to estimate performance indicators in the presence of uncontrollable measures, changes in performance measures related to network processes with uncontrollable indexes and price data have not been investigated.

Therefore, this study considers each DMU as a two-stage network, which includes uncontrollable inputs in the second stage. Given that the price of inputs of the first stage (initial input) and outputs of the second stage (final output) are available, models are presented to calculate the relative and cost efficiencies. Then, the inputs of the first stage are estimated using the generalized IDEA by perturbing the final outputs so that the technical and cost efficiencies remain unchanged. Using another presented model by perturbing the inputs of the first stage, final outputs are estimated while maintaining technical and revenue efficiencies. The proposed algorithms are then applied to a numerical example and a dataset related to 17 salmon farms in five western provinces of Iran. Moreover, the proposed method is compared with one of the existing methods.

Section 2 presents methods to estimate the performance measures of network systems with uncontrollable inputs and price information. Datasets are presented in Section 3 to demonstrate the introduced approach. Finally, Section 4 is allocated to the results and suggestions.

2. Inverse cost and revenue efficiencies for networks with uncontrollable inputs

Suppose there are n DMUs, where each unit has a two-stage network structure with uncontrollable inputs in the second stage. That is, DMU_o is in the form shown in Figure 1: Here, $x_{io},\ (i = 1, 2, ..., m)$ and $z_{ko},\ (k = 1, 2, ..., t)$ are the inputs of the first stage and the intermediate measures, respectively. The index z_{ko} is considered the output of the first stage and the input of the second stage. In the second stage, $d_{lo},\ (l = 1, 2, ..., g)$ and $y_{ro},\ (r = 1, 2, ..., s)$ show uncontrollable inputs and the final outputs, respectively.

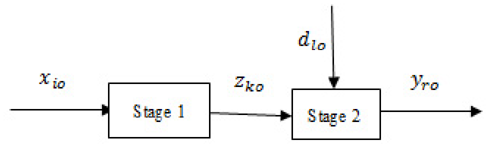

Figure 1: *The two-stage network structure.*

Accordingly, the technical efficiency can be obtained from the following model:

$$\begin{array}{lll} & \theta = \min\ \theta_o & \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_o x_{io}, & i = 1, \ldots, m, \\ & \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \geq 0, & k = 1, \ldots, t, \\ & \sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, & l = 1, \ldots, g, \\ & \sum_{j=1}^{n} \mu_j y_{rj} \geq y_{ro}, & r = 1, \ldots, s, \\ & \lambda_j, \mu_j \geq 0, & j = 1, \ldots, n. \end{array} \tag{1}$$

Suppose $(\theta^*, \lambda^*, \mu^*)$ is the optimal solution for model (1), with θ^* indicating the technical efficiency of DMU_o. If $\theta^* = 1$, then DMU_o is efficient; otherwise, it is inefficient. If the cost or price of the inputs of the first stage is available, the cost efficiency for DMU_o can be obtained. Let $c \in R^m$ be the cost or price of the inputs of the first stage. The minimum cost related to DMU_o is calculated using the following model:

$$\begin{array}{lll} \min & \sum_{i=1}^{m} c_i x_i & \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, & i = 1, \ldots, m, \\ & \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \geq 0, & k = 1, \ldots, t, \\ & \sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, & l = 1, \ldots, g, \\ & \sum_{j=1}^{n} \mu_j y_{rj} \geq y_{ro}, & r = 1, \ldots, s, \\ & \lambda_j, \mu_j \geq 0, & j = 1, \ldots, n, \\ & x_i \geq 0, & i = 1, \ldots, m. \end{array} \tag{2}$$

Definition 1. Suppose (x^*, λ^*, μ^*) is the optimal solution of model (2). The cost efficiency is defined as the ratio of the minimum cost $(c^t x^*)$ to the actual cost of DMU_o $(c^t x_o)$, i.e.,

$$CE_o = \frac{c^t x^*}{c^t x_o} = \frac{\sum_{i=1}^{m} c_i x_i^*}{\sum_{i=1}^{m} c_i x_{io}} \tag{3}$$

2.1. Minimum inverse cost for two-stage networks with uncontrollable inputs

If the prices of the inputs related to the first stage are available, they can also be used in IDEA models. Now suppose that in two-stage networks, where uncontrollable inputs are presented in the second stage, the final outputs of the second stage are perturbed from the level y_o to $y_o + h_o$. The following multi-objective model is proposed to estimate the inputs of the first stage for this perturbation, considering unchanged technical and cost efficiencies:

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m} c_i \bar{x}_i & \\
\min & (\alpha_1, \alpha_2, \ldots, \alpha_m) & \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \bar{x}_i, & i = 1, \ldots, m \quad (4.1), \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_o^* \alpha_i, & i = 1, \ldots, m \quad (4.2), \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \geq 0, & k = 1, \ldots, t \quad (4.3), \\
& \sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, & l = 1, \ldots, g \quad (4.4), \\
& \sum_{j=1}^{n} \mu_j y_{rj} \geq y_{ro} + h_{ro}, & r = 1, \ldots, s \quad (4.5), \\
& \sum_{i=1}^{m} c_i \bar{x}_i = \mathrm{CE}_o \sum_{i=1}^{m} c_i \alpha_i \quad (4.6), & \\
& \lambda_j, \mu_j \geq 0, & j = 1, \ldots, n, \\
& \bar{x}_i, \alpha_i \geq 0, & i = 1, \ldots, m.
\end{array}
\tag{4}
$$

According to the statement (4.6), we have:

$$
\min \sum_{i=1}^{m} c_i \overline{x}_i = \min CE_o \sum_{i=1}^{m} c_i \alpha_i
$$

Because the cost efficiency CE_o calculated using equation (3) is a numerical value, it is sufficient to calculate $min \sum_{i=1}^{m} c_i \alpha_i$. In this way, the multi-objective model (4) becomes as the following linear programming (LP) model:

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m} c_i \alpha_i & \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \bar{x}_i, & i = 1, \ldots, m \quad (5.1) \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_o^* \alpha_i, & i = 1, \ldots, m \quad (5.2) \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \geq 0, & k = 1, \ldots, t \quad (5.3) \\
& \sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, & l = 1, \ldots, g \quad (5.4) \\
& \sum_{j=1}^{n} \mu_j y_{rj} \geq y_{ro} + h_{ro}, & r = 1, \ldots, s \quad (5.5) \\
& \sum_{i=1}^{m} c_i \bar{x}_i = CE_o \sum_{i=1}^{m} c_i \alpha_i \quad (5.6) & \\
& \lambda_j, \mu_j \geq 0, & j = 1, \ldots, n \\
& \bar{x}_i, \alpha_i \geq 0, & i = 1, \ldots, m
\end{array}
\tag{5}
$$

Here, CE_o is the cost efficiency calculated using Eq. (3).

Definition 2. Let $(\lambda, \mu, \alpha, \ \overline{x})$ is a feasible solution for model (5). $(\lambda, \mu, \alpha, \ \overline{x})$ is a weak efficient solution of model (5) if there is no feasible solution $(\lambda', \mu', \alpha', \ \overline{x}')$ such that $\alpha_i' \leq \alpha_i \ (\forall i)$.

Theorem 1. *Suppose DMU_o is a two-stage network with uncontrollable inputs in the second stage. Let θ_o^* be the optimal objective of model (1). If $(\lambda, \mu, \alpha, \ \overline{x})$ is the weak efficiency solution for model (5), then the technical and cost efficiencies of DMU_o remain unchanged.*

Proof. Consider the following problem:

$$
\begin{aligned}
\min \quad & \bar{\theta} \\
\text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leq \bar{\theta}\alpha_i, && i=1,\dots,m \\
& \textstyle\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \geq 0, && k=1,\dots,t \\
& \textstyle\sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, && l=1,\dots,g \\
& \textstyle\sum_{j=1}^{n} \mu_j y_{rj} \geq y_{ro} + h_{ro}, && r=1,\dots,s \\
& \lambda_j, \mu_j \geq 0, && j=1,\dots,n
\end{aligned}
\tag{6}
$$

Suppose $\overline{\theta}^*$ is the optimal objective value of model (6), It is necessary to show that $\overline{\theta}^* = \theta_o^*$. Because $(\lambda, \mu, \alpha,\ \overline{x})$ is a weak efficiency solution for model (5), it applies to all constraints of model (5), including the following conditions:

$$
\begin{aligned}
& \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_o^* \alpha_i, && \forall i, \\
& \textstyle\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \geq 0, && \forall k, \\
& \textstyle\sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, && \forall l, \\
& \textstyle\sum_{j=1}^{n} \mu_j y_{rj} \geq y_{ro} + h_{ro}, && \forall r.
\end{aligned}
$$

As a result, $(\lambda\ ,\ \mu\ ,\ \theta_o^*)$ is a feasible solution to problem (6), so $\overline{\theta}^* \leq \theta_o^*$ (Eqs. (5.1) and (5.6) of model (5) are also established). Now it is enough to show that $\overline{\theta}^* \not< \theta_o^*$. Suppose $\overline{\theta}^* < \theta_o^*$, then it can be assumed $\overline{\theta}^* = k\theta_o^*$ that $0 < k < 1$. Thus, according to the first constraint of problem (6), we have:

$$
\begin{aligned}
& \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leq k\theta_o^* \alpha_i, && \forall i, \\
& \textstyle\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \geq 0, && \forall k, \\
& \textstyle\sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, && \forall l, \\
& \textstyle\sum_{j=1}^{n} \mu_j y_{rj} \geq y_{ro} + h_{ro}, && \forall r.
\end{aligned}
$$

As a result, $(\lambda, \mu, k\alpha,\ \overline{x})$ is a feasible solution for model (5) that $0 < k < 1$, which contradicts the weak efficient solution of $(\lambda, \mu, \alpha,\ \overline{x})$ for model (5). Hence, $\overline{\theta}^* \not< \theta_o^*$ leads to $\overline{\theta}^* = \theta_o^*$. That is, technical efficiency does not change.

For cost efficiency, according to the constraint (5.6) of model (5), we have $\sum_{i=1}^{m} c_i \overline{x}_i = CE_o \sum_{i=1}^{m} c_i \alpha_i$ that leads to $\frac{\sum_{i=1}^{m} c_i \overline{x}_i}{\sum_{i=1}^{m} c_i \alpha_i} = CE_o$, where CE_o is the cost efficiency related to DMU_o. By perturbing the final output from the level of y_{ro} to $y_{ro} + h_{ro}$, inputs can also change from the level of x_{io} to $\alpha_i = x_{io} + \Delta x_{io}$. Considering $\sum_{i=1}^{m} c_i \overline{x}_i$ as the minimum cost after perturbing the final outputs, $\sum_{i=1}^{m} c_i \alpha_i$ is the actual cost of the unit after perturbation. In other words, $\frac{\sum_{i=1}^{m} c_i \overline{x}_i}{\sum_{i=1}^{m} c_i \alpha_i}$ is the cost efficiency of the new DMU_o with the changed inputs and outputs of $(x_o + \Delta x_o,\ y_o + h_o)$, which is equal to the cost efficiency of the initial DMU_o (equal to CE_o). As a consequence, cost efficiency will not change.

According to IDEA models, when inputs/outputs are perturbed, the outputs/inputs may also change. In two-stage networks with uncontrollable inputs in the second stage, the minimum value of the network's initial inputs by perturbing the outputs is estimated while keeping technical and cost efficiencies unchanged. For this purpose, the following algorithm (cost algorithm) is presented using models (2), (3), and (5). The steps of the proposed algorithm are shown in Figure 2.

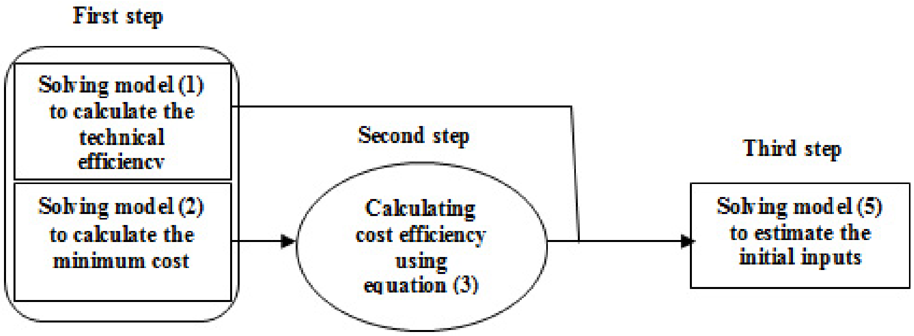

Figure 2: *The proposed algorithm for the input estimation of the initial stage with uncontrollable inputs.*

2.2. Output-oriented technical and revenue efficiencies for two-stage networks with uncontrollable inputs

Suppose there are n DMUs in the frame of two-stage networks with uncontrollable inputs as shown in Figure (1). The output-oriented efficiency of DMU_o is calculated using the following model:

$$\begin{array}{lll} & \varphi = \max\ \varphi_o & \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io}, & i = 1, \dots, m, \\ & \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \ge 0, & k = 1, \dots, t, \\ & \sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, & l = 1, \dots, g, \\ & \sum_{j=1}^{n} \mu_j y_{rj} \ge \varphi_o y_{ro}, & r = 1, \dots, s, \\ & \lambda_j, \mu_j \ge 0, & j = 1, \dots, n. \end{array} \tag{7}$$

Let $(\varphi^*, \lambda^*, \mu^*)$ be the optimal solution for model (7), with φ^* indicating the output-oriented technical efficiency of DMU_o. If $\varphi^* = 1$, then DMU_o is efficient; otherwise, it is inefficient.

If the output prices of the second stage are available, the revenue efficiency for DMU_o can be obtained. Let $p \in R^s$ be the output price of the second stage. The maximum revenue of DMU_o is calculated using the following model:

$$\begin{array}{lll} \max & \sum_{r=1}^{s} p_r y_r & \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io}, & i = 1, \dots, m \\ & \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \ge 0, & k = 1, \dots, t \\ & \sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, & l = 1, \dots, g \\ & \sum_{j=1}^{n} \mu_j y_{rj} \ge y_r, & r = 1, \dots, s \\ & \lambda_j, \mu_j \ge 0, & j = 1, \dots, n \\ & y_r \ge 0, & r = 1, \dots, s. \end{array} \tag{8}$$

Definition 3. Let (y^*, λ^*, μ^*) are the optimal solution for model (8). The ratio of real revenue $(p^t y_o)$ to the maximum revenue of DMU_o $(p^t y^*)$ shows the revenue efficiency.

$$RE_o = \frac{p^t y_o}{p^t y^*} = \frac{\sum_{r=1}^{s} p_r y_{ro}}{\sum_{r=1}^{s} p_r y_r^*} \tag{9}$$

2.3. Maximum inverse revenue for two-stage networks with uncontrollable inputs

Given the availability of the output prices of the second stage, this information can also be used in IDEA models. Suppose that in a two-stage network with uncontrollable inputs in the second stage, the inputs of the first stage are perturbed from the level x_o to $x_o + b_o$. The following multi-objective model is proposed to estimate the outputs of the second stage while keeping the output-oriented technical and revenue efficiencies unchanged:

$$
\begin{array}{lll}
\max & \sum_{r=1}^{s} p_r \bar{y}_r & \\
\max & (\beta_1,\ \beta_2,\ \ldots,\ \beta_s) & \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} + b_{io}, & i = 1, \ldots, m \quad (10.1) \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \geq 0, & k = 1, \ldots, t \quad (10.2) \\
& \sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, & l = 1, \ldots, g \quad (10.3) \\
& \sum_{j=1}^{n} \mu_j y_{rj} \geq \bar{y}_r, & r = 1, \ldots, s \quad (10.4) \\
& \sum_{j=1}^{n} \mu_j y_{rj} \geq \varphi_o^* \beta_r, & r = 1, \ldots, s \quad (10.5) \\
& \sum_{r=1}^{s} p_r \bar{y}_r = \frac{\sum_{r=1}^{s} p_r \beta_r}{RE_o} \quad (10.6) & \\
& \lambda_j, \mu_j \geq 0, & j = 1, \ldots, n \\
& \bar{y}_r, \beta_r \geq 0, & r = 1, \ldots, s
\end{array} \tag{10}
$$

According to the statement (10.6), we have:

$$
\max \sum_{r=1}^{s} p_r \bar{y}_r = \max \frac{\sum_{r=1}^{s} p_r \beta_r}{RE_o}
$$

Here, $\frac{1}{RE_o}$ is the inverse of the revenue efficiency calculated by equation (9), representing a numerical value. Therefore, it is enough to calculate $max\ \sum_{r=1}^{s} p_r \beta_r$. In this way, the multi-objective model (10) becomes the following LP model:

$$
\begin{array}{lll}
\max & \sum_{r=1}^{s} p_r \beta_r & \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} + b_{io}, & i = 1, \ldots, m, \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{kj} \geq 0, & k = 1, \ldots, t, \\
& \sum_{j=1}^{n} \mu_j d_{lj} = d_{lo}, & l = 1, \ldots, g, \\
& \sum_{j=1}^{n} \mu_j y_{rj} \geq \bar{y}_r, & r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \mu_j y_{rj} \geq \varphi_o^* \beta_r, & r = 1, \ldots, s, \\
& \sum_{r=1}^{s} p_r \bar{y}_r = \frac{\sum_{r=1}^{s} p_r \beta_r}{RE_o}, & \\
& \lambda_j, \mu_j \geq 0, & j = 1, \ldots, n, \\
& \bar{y}_r, \beta_r \geq 0, & r = 1, \ldots, s.
\end{array} \tag{11}
$$

Here, RE_o is the revenue efficiency calculated using equation (9).

Definition 4. Let $(\lambda, \mu, \beta,\ \overline{y})$ is a feasible solution for model (11). Then, $(\lambda, \mu, \beta,\ \overline{y})$ is a weak efficient solution for model (11) provided that there is no feasible solution $(\lambda', \mu', \beta',\ \overline{y}')$ such that $\beta'_r \geq \beta_r \ \ (\forall r)$.

Theorem 2. *Suppose DMU_o is a two-stage network with uncontrollable inputs in the second stage. Let φ_o^* is the optimal objective of model* (7)*. If $(\lambda, \mu, \beta, \ \overline{y})$ is the weak efficiency solution for model* (11)*, the output-oriented technical and revenue efficiencies of DMU_o remain unchanged.*

Proof. The proof is similar to Theorem 1.

In network structures with uncontrollable inputs in the second stage, perturbing the inputs of the first stage can change the final outputs. The following proposed (revenue) algorithm is used to estimate the maximum output of the second stage of the network while keeping both output-oriented and the revenue efficiencies unchanged.

Algorithm 2: Inverse revenue algorithm for networks with uncontrollable inputs

Step 1) Calculate the output-oriented efficiency of the units using model (7).

Step 2) Calculate the maximum revenue using model (8) and then calculate the revenue efficiency using equation (9).

Step 3) Apply model (11) to estimate the maximum value of the final outputs by perturbing the inputs of the first. □

2.4. Numerical Example

Consider 10 DMUs with a two-stage network structure shown below:

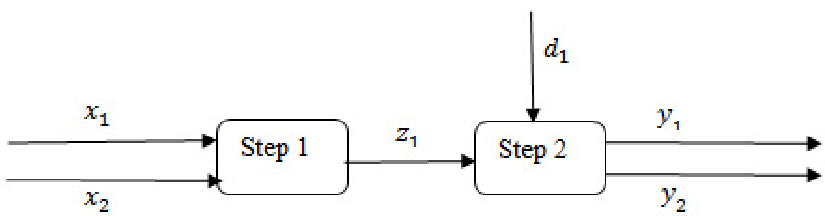

Figure 3: *The two-stage network structure.*

The inputs and outputs of these 10 units are given in Table 1.

	x_1	x_2	z_1	d_1	y_1	y_2
DMU_1	6.2	62.4	58	8.1	51.1	1.64
DMU_2	4.1	47.4	88	8.1	33.3	1.74
DMU_3	10.0	100.0	99	6.3	100.0	9.99
DMU_4	2.8	44.9	25	10.0	31.4	2.69
DMU_5	5.5	51.9	13	8.1	37.5	2.27
DMU_6	5.0	60.6	48	8.1	45.8	4.18
DMU_7	3.1	39.6	73	9.1	34.9	0.15
DMU_8	3.0	35.4	67	10.0	26.3	1.25
DMU_9	2.9	38.0	86	8.1	29.1	1.43
DMU_{10}	6.1	70.8	48	6.3	72.4	6.02

Table 1: *Inputs and outputs of 10 networks*

According to the cost algorithm, we first obtain the technical efficiency of the units using model (1) and then calculate the cost efficiency considering the input prices ($c_1 = 2$ and $c_2 = 3$), model (2), and equation (3). Then, we estimate the minimum amount of inputs using model (5) by perturbing the first output by 10% and the second output by 20%. The estimation results of the inputs related to the first stage and the percentage of changes are given in Table 2.

	New x_1	Percentage of changes x_1	New x_2	Percentage of changes x_2
DMU_1	5.62	-8.8%	73.61	17.9%
DMU_2	3.62	-11.1%	47.4	-0.1%
DMU_3	9.61	-4.1%	125.93	25.9%
DMU_4	3.46	25.1%	45.35	1%
DMU_5	5.19	-6%	68.04	31.2%
DMU_6	5.77	14.6%	75.63	24.8%
DMU_7	3.02	-3.3%	39.6	-0.1%
DMU_8	2.7	-11.5%	35.4	0.1%
DMU_9	2.9	1.2%	38	0.1%
DMU_{10}	7.14	17.8%	93.61	32.2%

Table 2: *Estimated inputs of the first stage and their percentage of changes*

According to Table 2, the first input decreases for units 1, 2, 3, 5, 7, and 8 and increases for other units. The largest decrease in the first input is related to DMU 8, with an 11.5% decrease, and the largest increase in this input is observed in DMU 4 by about 25%. Also, the second input of DMUs 2 and 7 decreases by approximately 0.1% but increases for other units. The largest increase in this input is related to unit 10 by about 32%.

We use the method presented by Shiri et al. [18] to compare the proposed method with existing studies, highlighting the following differences between the proposed technique and Shiri et al.'s model:

- Shiri et al. [18] use four models to solve such problems. They first calculate technical efficiency and cost efficiency using two models, then use two other models (the first has three and the second has four types of constraints) to estimate the minimum input while maintaining technical efficiency and cost efficiency. After calculating technical and cost efficiencies, the proposed model would reach the same results using a model with only one additional constraint. Accordingly, results confirm that the proposed method would reduce the number of computational operations and the time required for calculations.
- Another difference is that there are uncontrollable inputs in the proposed network, which have not been taken into account by Shiri et al.'s method.

For comparisons, we consider the uncontrollable input index in Shiri et al.'s method, adding the third constraint of model (11) to the models of Shiri et al. [18]. The results are presented in Table 3.

According to Table 3, the first input decreases for units 1, 2, 3, 5, and 8 and increases for other units. The largest decrease is 8.8%, associated with DMU 1, and the largest increase is 30.5%, related to DMU 4. Also, the second input values of all units increase, with the smallest and largest increase reported for units 4 and 10, respectively. A comparison of Tables 2 and 3 reveals that the inputs of some units decrease in the proposed method but increase in the method of Shiri et al. (such as the first input of DMU 7 and the second input of units 2 and 7). The changes in some units are the same in both methods.

2.5. Case Study

The data are related to the performance of 17 salmon farms in five western provinces of Iran in 2018 (Abdali [1]). The framework under examination of fish farms is shown in Fig. 4, considering each salmon farm as a two-stage network system. The first stage is the baby fish's production pool, and the second stage is the adult fish's breeding pool. First, all the fish are

	New x_1	Percentage of changes x_1	New x_2	Percentage of changes x_2
DMU_1	5.62	-8.8%	73.61	17.9%
DMU_2	3.98	-2.3%	52.14	9.9%
DMU_3	9.61	-4.1%	125.93	25.9%
DMU_4	3.61	30.5%	47.25	5.2%
DMU_5	5.19	-6%	68.04	31.2%
DMU_6	5.77	14.6%	75.63	24.8%
DMU_7	3.32	6.3%	43.56	9.9%
DMU_8	2.97	-2.6%	38.94	10.1%
DMU_9	3.19	11.4%	41.8	10.1%
DMU_{10}	7.14	17.8%	93.61	32.2%

Table 3: *Results of Shiri et al.'s method considering the uncontrollable input*

kept in the first pool for spawning, and then the baby fish are sent to the second pool to grow up. The final output is the adult fish for human consumption. The cost of ecosystem damage is the uncontrollable input in the second pool, where the young fish are placed to grow up. The terms used are defined as follows:

x_i: Value of fish brought into the first pool and kept in that pool for spawning (unit of 1000 kg).

d_l: Costs of ecosystem damage in the second pool of the salmon farm (unit of one million Tomans)

z_k: The number of baby fish sent to the second pool to grow up (unit of 1000 pieces).

y_r: Value of mature fish for people's food consumption (unit of 1000 kg).

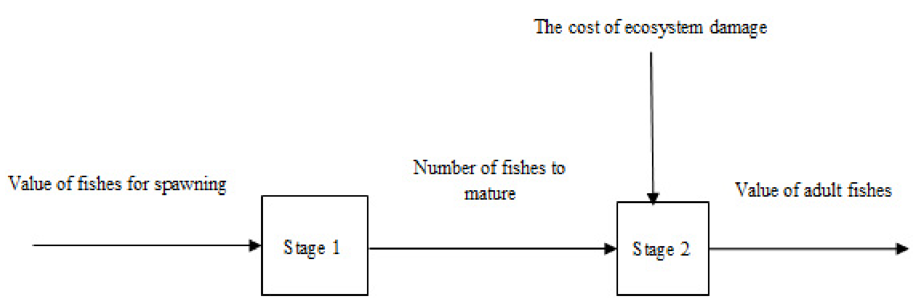

Figure 4: *The two-stage process related to fish farms.*

Table 4 shows the inputs and outputs of 17 salmon farms in five western provinces of Iran. Given the cost algorithm and the price of the inputs of the first stage (c=5), the cost efficiency of the units is determined using model (2) and Expression (3) after determining the technical efficiency (using model (1)). Then, the output of the second stage (amount of mature fish for consumption) is perturbed by 10% to estimate the minimum amount of fish for spawning after this perturbation using model (5) so that technical and cost efficiencies remain unchanged. The results of the amount of fish for spawning, the percentage of changes, the technical efficiency, and the cost efficiency of the units are given in Table 5.

Stage 1			Stage 2	
DMU_j	x_i	z_k	d_l	y_r
DMU_1	656	98.541	56	3564
DMU_2	5356	32.654	29	2145
DMU_3	354	82.365	67	3698
DMU_4	542	24.563	45	5246
DMU_5	652	98.543	19	7145
DMU_6	752	54.263	45	3369
DMU_7	326	65.478	36	5214
DMU_8	542	36.985	68	5369
DMU_9	632	45.638	45	5412
DMU_{10}	623	96.353	36	2148
DMU_{11}	354	65.878	45	4532
DMU_{12}	654	36.546	42	4521
DMU_{13}	423	54.692	25	3678
DMU_{14}	362	24.578	42	4213
DMU_{15}	452	36.925	36	2846
DMU_{16}	325	57.892	24	6532
DMU_{17}	542	65.245	45	4215

Table 4: *Inputs and outputs of 17 salmon farms*

DMU_j	*New* x_i	Percentage of changes	Technical efficiency	Cost efficiency
DMU_1	656	0%	0.1996	0.1996
DMU_2	5357.23	0.02%	0.0127	0.0127
DMU_3	354	0%	0.4424	0.4424
DMU_4	680.85	25.62%	0.1948	0.1948
DMU_5	778.25	19.36%	0.6496	0.6496
DMU_6	752.24	0.03%	0.1399	0.1399
DMU_7	390.01	19.63%	0.4199	0.4199
DMU_8	542.41	0.08%	0.2933	0.2933
DMU_9	786.51	24.45%	0.1806	0.1806
DMU_{10}	623	0%	0.1351	0.1351
DMU_{11}	354.34	0.1%	0.2978	0.2978
DMU_{12}	680.87	4.11%	0.1505	0.1505
DMU_{13}	504.83	19.35%	0.2317	0.2317
DMU_{14}	362.34	0.09%	0.2718	0.2718
DMU_{15}	452.34	0.08%	0.1862	0.1862
DMU_{16}	422.93	30.13%	0.7656	0.7656
DMU_{17}	542.48	0.09%	0.1944	0.1944

Table 5: *Findings of performance*

According to the results of Table 5, the amount of fish for spawning has not changed in units 1, 3, and 10 (0% change), while the amount of fish for spawning has a slight change in units 2, 6, 8, 11, 14, 15, and 17 (increased by less than one-tenth percent). Units 16, 4, and 9 have undergone the greatest changes, by about 30%, 26%, and 24% increase, respectively.

The fourth and fifth columns of Table 5 show the technical and cost efficiencies of the units, respectively. Notice that each unit has only one input and one output, leading to the equivalence of technical and cost efficiencies [20].

Using the revenue algorithm, we first obtain the output-oriented technical efficiency and the revenue efficiency of the units using models (7) and (8) and the relation (9) (we consider the output price p=10). Then, the amount of fish for spawning is perturbed by 10%, and the maximum amount of final output is estimated for this perturbation using model (11) so that the output-oriented technical and revenue efficiencies remain unchanged. The new output results, the percentage of their changes, the output-oriented technical efficiency, and the revenue efficiency of the units are given in Table 6.

DMU_j	*New* y_r	Percentage of changes	Output-oriented efficiency	Revenue efficiency
DMU_1	3692.12	3.59%	4.4609	0.2242
DMU_2	2145	0%	5.0842	0.1967
DMU_3	3916.37	5.91%	3.1445	0.3180
DMU_4	5437.96	3.66%	2.4600	0.4065
DMU_5	7145	0%	1.0000	1.0000
DMU_6	3518.8	4.45%	4.3736	0.2286
DMU_7	5585.2	7.12%	1.7035	0.5870
DMU_8	5736.81	6.85%	2.8583	0.3499
DMU_9	5629.69	4.02%	2.5294	0.3954
DMU_{10}	2245.37	4.53%	5.5746	0.1794
DMU_{11}	4841.19	6.82%	2.2208	0.4503
DMU_{12}	4714.18	4.27%	2.9496	0.3390
DMU_{13}	3842.67	4.48%	2.2380	0.4468
DMU_{14}	4508.62	7.02%	2.3752	0.4210
DMU_{15}	2952.9	3.76%	3.6839	0.2715
DMU_{16}	6789.04	3.94%	1.1016	0.9078
DMU_{17}	4369.23	3.66%	3.0617	0.3266

Table 6: *Efficiencies, new outputs, and changes*

According to Table 6, the amount of adult fish has not changed in units 2 and 5 (0% changes), and the amount of adult fish has changed by 7.12%, 7.02%, and 6.85% in units 7, 14, and 8, respectively. Unit 11 shows the highest increase by 6.82%. Output-oriented technical and revenue efficiencies are provided in the fourth and fifth columns, revealing that unit 5 has both output-oriented technical and revenue efficiencies.

Uncontrollable indicators in network structures are important and effective in finding accurate results. Using the proposed models, it is possible to examine the changes of indicators in network processes with uncontrollable inputs while the prices of some indicators are known. For more explanation, the methodology employed allows for the efficiency evaluation of DMUs and performance measures estimation within a two-stage network framework. By distinguishing between the controllable inputs in the first stage and the uncontrollable inputs in the second stage, we can gain deeper insights into overall performance. Businesses can utilize the inverse network models to estimate the necessary adjustments to inputs and outputs without altering cost/revenue and technical efficiencies. This allows managers to develop targeted strategies. Understanding the efficiency metrics can guide decision-makers in resource reallocation within the network with uncontrollable inputs. Regularly process assessment and recalibration based on efficiency metrics enables businesses to adapt to changing conditions and maintain competitive advantage.

In summary, the results of our study provide a robust framework for organizations to diagnose inefficiencies, implement targeted improvements, and ultimately enhance their business processes.

3. Conclusions

DEA evaluates the relative efficiency of a set of DMUs using mathematical programming models. In IDEA, the required inputs (outputs) are estimated by perturbing outputs (inputs) while maintaining the efficiency value. On the other hand, if input or output prices are available, the cost or revenue efficiencies of DMUs can be obtained using DEA. This paper has considered each DMU a two-stage network with uncontrollable inputs in the second stage. First, according to the availability of the input cost of the first and the output price of the second stage, models have been proposed to calculate the relative, cost, and revenue efficiencies of the desired network structures. Then, using the introduced inverse network models, inputs/outputs are estimated, while the cost/revenue efficiency remains unchanged in addition to the technical efficiency. Finally, the proposed algorithms are applied to a numerical example and data related to the performance information of 17 salmon farms in five western provinces of Iran. Also, the proposed method is compared with one of the existing methods, confirming that it is useful and practical for estimating performance indicators in network systems with uncontrollable inputs.

While the presented approach effectively addresses two-stage systems with precise data, it is important to acknowledge several key limitations. Our method assumes that data are precise, which may not always be the case in real-world applications. Future research could explore the robustness of our approach when dealing with imprecise or uncertain data, potentially incorporating techniques from fuzzy logic or stochastic modeling. Furthermore, the current method is tailored to specific network structures. Developing adaptive techniques that can cater to a wider variety of network configurations would enhance the versatility of our approach. Future work could focus on generalizing our method to accommodate different types of networks, including dynamic structures.

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