# Numerical methods for checking the stability of gyroscopic systems

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**Abstract.** Gyroscopic mechanical systems are modeled by the second-order differential equation

$$M\ddot{x}(t) + G\dot{x}(t) + Kx(t) = 0,$$

where $M \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix, $G \in \mathbb{R}^{n\times n}$ is a skew-symmetric ($G^T = -G$) matrix, and $K \in \mathbb{R}^{n\times n}$ is a symmetric matrix, representing the mass, gyroscopic, and stiffness matrices, respectively. The stability of such systems, which is the primary topic of this paper, is determined by the properties of the associated quadratic eigenvalue problem (QEP)

$$\mathcal{G}(\lambda)x = (\lambda^2 M + \lambda G + K)x = 0, \quad x \in \mathbb{C}^n,\ x \neq 0.$$

In this paper, we provide an overview of various linearizations of the QEP and propose numerical methods for checking the stability of gyroscopic systems based on solving the linearized problem. We present examples that demonstrate how the use of numerical methods provides a significantly larger stability region, which cannot be detected using the considered non-spectral criteria, or verify stability in cases where non-spectral criteria are not applicable, highlighting the advantages of numerical methods.

**Keywords**: generalized eigenvalues, gyroscopic systems, linearization, quadratic eigenvalue problem, stability

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## 1. Introduction

Gyroscopic mechanical systems originate from the study of rotating bodies and their dynamic behaviour. They play a crucial role in various disciplines, such as physics, engineering, and applied mathematics.

The dynamics of gyroscopic systems oscillating about an equilibrium position, with no external forces applied, is mathematically modelled by the equation

$$M\ddot{x}(t) + G\dot{x}(t) + Kx(t) = 0, \tag{1}$$

where the mass matrix $M \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, the gyroscopic matrix $G \in \mathbb{R}^{n\times n}$, which represents the effect of gyroscopic internal forces, is skew-symmetric ($G^T = -G$), and the stiffness matrix $K \in \mathbb{R}^{n\times n}$ is symmetric. Here, $x = x(t)$ represents the time-dependent displacement vector.

The solutions of the equation (1) can be expressed in terms of the eigenvalues and eigenvectors of the quadratic matrix polynomial

$$\mathcal{G}(\lambda) = \lambda^2 M + \lambda G + K, \tag{2}$$

so the behaviour of the gyroscopic system (1) is fully characterized by the algebraic properties of the quadratic matrix polynomial (2) and the associated quadratic eigenvalue problem (QEP)

$$\mathcal{G}(\lambda)x = (\lambda^2 M + \lambda G + K)x = 0, \quad x \in \mathbb{C}^n, \quad x \neq 0. \tag{3}$$

One of the most important properties of mechanical systems, which is also the main topic of this paper, is stability. For gyroscopic systems, stability is achieved when the norm $\|x(t)\|$ of the solution of equation (1) is uniformly bounded for all $t > 0$ (see, for example, [1]). This condition is satisfied if and only if all eigenvalues of the quadratic matrix polynomial $\mathcal{G}$ are purely imaginary and semisimple, i.e., have equal geometric and algebraic multiplicities (see [1, 3]). It is well known that for all positive definite $K$ (which physically corresponds to a system oscillating about a stable equilibrium point), the system (1) is stable. For negative definite $K$ (corresponding to oscillation about an unstable equilibrium point) or indefinite $K$, there are no simple necessary and sufficient conditions for stability. Although there are some analytical 'non-spectral' stability criteria, they are often too restrictive and cannot be generalized to different types of system matrices. Rather than relying solely on analytical conditions, this paper explores numerical methods for checking the stability. This computational approach overcomes the limitations of theoretical results, offering a more comprehensive solution to the stability problem in gyroscopic mechanical systems (see [5, 6, 12, 18]).

More precisely, in this paper, we provide an overview of various linearizations of the QEP (3) and numerical methods for checking the stability of gyroscopic systems, which are based on solving the linearized problem. These methods do not require any prior information about the interconnections between the system matrices or the definiteness of the matrix $K$, which is usually needed for non-spectral criteria. We present an example often used to compare analytical criteria for determining the stability region of parameter-dependent problems and demonstrate that numerical methods provide a significantly larger stability region, which cannot be detected using the considered analytical criteria. Furthermore, we present examples where analytical criteria are difficult or impossible to apply while numerical methods provide relevant results.

System stability is often a fundamental requirement for analyzing other system properties. For example, in our previous work, we focused on perturbation analysis for stable gyroscopic systems, assuming stability in advance. In paper [14], we studied perturbation bounds for gyroscopic systems in motion about a stable equilibrium position. In particular, we studied perturbations in the eigenvalues and eigenvectors of stable gyroscopic systems with symmetric and positive definite stiffness matrix $K$. In [15], we analyzed gyroscopic systems in motion about an unstable equilibrium position. These systems are modelled as linear gyroscopic mechanical systems of the form (1), where the stiffness matrix $K$ is symmetric and negative definite. In both papers, we derived new relative bounds for the perturbation of the eigenvalues and bounds of the $\sin\Theta$ - type for the associated eigenspaces based on results from [20].

The rest of the paper has the following structure: Section 2 provides a brief introduction to the stability of gyroscopic systems and presents related results used later in the paper. Section 3 examines the stability of gyroscopic systems in terms of the generalized eigenvalues of matrix pencils obtained through several different types of linearizations. Section 4 presents numerical methods for calculating system eigenvalues and their multiplicities. Section 5 includes numerical examples, and Section 6 provides an overview of the results.

Notation. Throughout this paper, $I_n$, $n \in \mathbb{N}$, denotes the $n \times n$ identity matrix. $\lambda(A)$ denotes the set of all eigenvalues of a matrix $A$. $A > 0$ ($A < 0$) denotes that the matrix $A$ is positive (negative) definite, while $A \geq 0$ ($A \leq 0$) denotes that $A$ is positive (negative) semidefinite.

## 2. Stability of gyroscopic systems

One of the most significant properties of the spectrum of the quadratic matrix polynomial $\mathcal{G}$ from (2) is its Hamiltonian symmetry, which means that it is symmetric with respect to both the real and imaginary axes (since $M$ is nonsingular, the spectrum of (2) is always a bounded set in the complex plane, meaning that (2) has $2n$ finite eigenvalues). Specifically, if $\lambda$ is an eigenvalue of $\mathcal{G}(\lambda)$, then $\overline{\lambda}$, $-\lambda$, and $-\overline{\lambda}$ are also eigenvalues, with associated pairs of left and right eigenvectors $(y, x)$, $(\overline{y}, \overline{x})$, $(x, y)$ and $(\overline{x}, \overline{y})$, respectively. Because of this, gyroscopic systems can only have the property of marginal stability, meaning they are considered stable if all eigenvalues of (2) are purely imaginary and semisimple. Therefore, a direct approach to check the stability of the system involves calculating all the eigenvalues of the quadratic matrix polynomial and verifying their algebraic and geometric multiplicities.

There are some very useful 'non-spectral' results on the stability of gyroscopic systems, expressed in terms of the properties of system matrices or their interconnections. The most significant are the results expressed only in terms of the definiteness of the matrix $K$. As already mentioned, a system with a positive definite $K$ is always stable. On the other hand, if $K$ is positive semidefinite, only zero eigenvalues can be defective (that is, not semisimple), and $\mathcal{G}(\lambda)$ has a defective eigenvalue if and only if $K$ is singular and there exist nonzero vectors in $G(\operatorname{Ker} K) \cap \operatorname{Im} K$ (see [16, Theorem 2]). Furthermore, for systems with an indefinite $K$, it is known that if $\det K < 0$, then (1) is unstable (see [16, Theorem 3],[17, Theorem 6.3]). Also, result from [4] show that the system is stable if $K < 0$ and

$$4K - G^2 - \delta I > 0,$$

where $\delta = 2(\delta_1 - \delta_2)$, and $\delta_1$ and $\delta_2$ are the maximal and minimal eigenvalues of $-K$, respectively. Therefore, if information about the definiteness of the matrix $K$ is available, some of the previously mentioned criteria can serve as necessary or sufficient conditions for the stability or instability of the system.

There are also criteria based on other system matrices and their properties or interconnections. For example, it is well known that the system $\ddot{x} + G\dot{x} + K = 0$ is unstable if

$$4K - G^2 < 0. \tag{4}$$

Unfortunaly, the converse is not true in the general case. However, in the case where $K$ and $G$ commute,

$$4K - G^2 > 0 \tag{5}$$

is both a necessary and sufficient condition for stability (see [10]). On the other hand, the condition

$$K - GM^{-1}G < 0$$

always implies instability of system (1) (see [7]).

The main disadvantage of all these criteria is that they are limited to specific types of systems, and some are difficult to verify.

## 3. Stability and generalized eigenproblem

The main goal of this paper is to investigate numerical methods for checking the stability of gyroscopic systems of the form (1), including the calculation of the eigenvalues of (2). A common approach to solving the QEP is to first linearize it into a linear eigenvalue problem. One possible class of linearizations of $\mathcal{G}$ is given by

$$L(\lambda) = \begin{bmatrix} 0 & X \\ -K & -G \end{bmatrix} - \lambda \begin{bmatrix} X & 0 \\ 0 & M \end{bmatrix}, \tag{6}$$

where $X \in \mathbb{R}^{n\times n}$ is an arbitrary regular matrix. Although the result is independent of $X$, the choice of suitable $X$ may provide advantages in numerical methods. Specifically, we can use linearization (6) with $X = I$, resulting in

$$L_M(\lambda) = \begin{bmatrix} 0 & I \\ -K & -G \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} := A - \lambda B\,. \tag{7}$$

The generalized eigenvalue problem of the matrix pencil $L_M$ and the QEP (3) are equivalent. This means that eigenvalues of $\mathcal{G}(\lambda)$ and the pencil $A - \lambda B$, i.e. the pair $(A, B)$ are the same and that the associated eigenvectors of one problem can be calculated from those of the other. Since $M$ is regular, instead of the generalized eigenvalues of the pair $(A, B)$, we can consider a standard eigenvalue problem $B^{-1}Ax = \lambda x$, which involves calculating the eigenvalues of the matrix

$$\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}G \end{bmatrix}. \tag{8}$$

If $M$ is very ill-conditioned, such that $M^{-1}K$ and $M^{-1}G$ cannot be computed accurately, the QEP should not be replaced with a standard eigenproblem and must instead be solved as a generalized eigenproblem. However, the structure of $M$ often allows for an efficient calculation of its inverse. Thus, if $M^{-1}$ is given, or, as is common in theoretical analyses, the mass matrix $M$ is assumed to be the identity matrix (see, for example, the criterion discussed in [1]), the standard eigenvalue problem with the matrix (8) may provide a more advantageous formulation. The main drawback of the linearization (7) is that the resulting eigenvalue problem is non-symmetric.

Linearization can also be chosen such that $A$ and $B$ preserve the special structures of the system matrices. There are some linearizations $A - \lambda B$ that reflect the property that the eigenvalues of a real gyroscopic system occur in quadruples $(\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda})$, where one of the matrices $A$ or $B$ is Hamiltonian, and the other is skew-Hamiltonian. The matrix $A$ is Hamiltonian if

$$(AJ)^T = AJ,$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

If

$$(AJ)^T = -AJ,$$

we say that $A$ is skew-Hamiltonian. An example of such a skew-Hamiltonian / Hamiltonian linearization is given by (see [19])

$$\begin{bmatrix} K & 0 \\ G & K \end{bmatrix} - \lambda \begin{bmatrix} 0 & K \\ -M & 0 \end{bmatrix}. \tag{9}$$

On the other hand, a Hamiltonian/skew-Hamiltonian linearization is

$$\begin{bmatrix} 0 & -K \\ M & 0 \end{bmatrix} - \lambda \begin{bmatrix} M & G \\ 0 & M \end{bmatrix}, \tag{10}$$

which is preferred when $K$ is singular (see [19]).

There are also linearizations that result in a Hermitian eigenvalue problem. To obtain such a linearization, from the matrix polynomial $\mathcal{G}$ from (2), we define a new matrix polynomial

$$\mathcal{H}(\lambda) = -\mathcal{G}(-i\lambda) = \lambda^2 M + \lambda H - K \tag{11}$$

where $i$ is the imaginary unit, and

$$H := iG$$

is a complex Hermitian matrix. The benefit of this transformation lies in the fact that all matrices in (11) are Hermitian, and the eigenvalues of $\mathcal{G}$ and $\mathcal{H}$ can simply be computed from each other. More specifically, if $\lambda$ is an eigenvalue of $\mathcal{G}$, then $i\lambda$ is an eigenvalue of $\mathcal{H}$, and conversely, both sharing the same eigenvector. As stated previously, the gyroscopic system (1) is stable if and only if all the eigenvalues of $\mathcal{G}$ are purely imaginary and semisimple. This equivalently means that all eigenvalues of $\mathcal{H}$ are real and semisimple. In the case where $K > 0$, the QEP associated with (11) is a hyperbolic eigenvalue problem, which is known to have real and semisimple eigenvalues. This is consistent with the statement that gyroscopic systems with $K > 0$ are stable (see [19]).

Due to the desirable properties of $\mathcal{H}(\lambda)$, instead of the original QEP (3), we can consider the QEP

$$\mathcal{H}(\lambda)x = 0, \quad x \in \mathbb{C}^n, \quad x \neq 0. \tag{12}$$

One possible Hermitian linearization of $\mathcal{H}$ is

$$L_{\mathcal{H}}(\lambda) = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} - \lambda \begin{bmatrix} H & M \\ M & 0 \end{bmatrix} := A - \lambda B\,. \tag{13}$$

The generalized eigenvalues of the original problem can be obtained from the eigenvalues of $A - \lambda B$ by multiplying them by $-i$. Since $A, B \in \mathbb{C}^{n\times n}$ are Hermitian matrices, the matrix pair $(A, B)$ is said to be Hermitian. Note that the linearizations discussed above involved real matrices, while in this case, $B$ is a complex matrix.

If $K < 0$, the linearization (13) can be further simplified. By transforming (13), one of the matrices in the pair $(A, B)$ can be reduced to a sign matrix $J$. Let $L_M$ and $L_C$ be the Cholesky factors of $M$ and $C = -K$ ($C$ is positive definite), such that $M = L_M L_M^*$ and $K = -C = -L_C L_C^*$. Instead of the pair $(A, B)$, we can consider the equivalent pair

$$(L^{-1}AL^{-*}, L^{-1}BL^{-*}) := (J, B_L),$$

where

$$L^{-1} = \begin{bmatrix} L_C^{-1} & 0 \\ 0 & L_M^{-1} \end{bmatrix}$$

and

$$J = \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix}, \quad B_L = \begin{bmatrix} L_C^{-1} H L_C^{-*} & L_C^{-1} L_M \\ L_M^* L_C^{-*} & 0 \end{bmatrix}. \tag{14}$$

This specific linearization was used in [15] to obtain relative perturbation bounds.

## 3.1. Stability and definite matrix pairs

A Hermitian matrix pair $(A, B)$ is said to be definite if there exists a real linear combination $\alpha A + \beta B$ of the matrices $A$ and $B$ that is positive definite. If either $A$ or $B$ is a definite matrix, the pair $(A, B)$ is trivially definite. Definite matrix pairs have real and semisimple eigenvalues. Therefore, if $(A, B)$ from (13) is definite, the corresponding gyroscopic system is stable (see [19]).

Using the ideas from [1, 8] applied to the QEP (12), we obtain that for the definiteness of the pair $(A, B)$ from (13), we have

$$\alpha A + \beta B = \begin{bmatrix} \alpha K + \beta H & \beta M \\ \beta M & \alpha M \end{bmatrix} = \tag{15}$$

$$= \begin{bmatrix} I & -\frac{\beta}{\alpha} I \\ 0 & I \end{bmatrix} \begin{bmatrix} \alpha K - \beta H - \frac{\beta^2}{\alpha} M & 0 \\ 0 & \alpha M \end{bmatrix} \begin{bmatrix} I & 0 \\ -\frac{\beta}{\alpha} & I \end{bmatrix}, \tag{16}$$

which implies that $\alpha A + \beta B$ is congruent to

$$\alpha \begin{bmatrix} -\mu^2 M - \mu H + K & 0 \\ 0 & M \end{bmatrix},$$

where $\mu = -\frac{\beta}{\alpha}$. Since $M$ is positive definite, the pair $(A, B)$ is definite if and only if

$$-\mu^2 M - \mu H + K$$

is positive definite for some $\mu$. Equivalently, this condition holds if and only if

$$\mu^2 M + \mu H - K$$

is negative definite for some $\mu$. Therefore, if we have information about the negative definiteness of this matrix, we can conclude that the system is stable. Notice that when $K$ is positive definite, then $A$ is also positive definite, and the pair $(A, B)$ is trivially definite. This corresponds to the well-known fact that gyroscopic systems with positive definite $K$ are always stable. However, while the positive definiteness of the pair $(A, B)$ implies that the system is stable, the converse is not true. A system can still be stable even if $(A, B)$ is not definite.

## 4. Numerical methods for detecting stability

In this section, we discuss numerical methods for detecting stability in gyroscopic systems. These methods involve the computation of eigenvalues of the pair $(A, B)$, which is derived from the gyroscopic system through the linearization of the corresponding QEP. One approach to obtaining the eigenvalues of $(A, B)$ is by reducing the pair to its generalized Schur form

$$Q^* A Z = S, \quad Q^* B Z = T, \tag{17}$$

where $Q, Z \in \mathbb{C}^{2n \times 2n}$ are unitary matrices, and $S, T \in \mathbb{C}^{2n \times 2n}$ are upper triangular. The generalized eigenvalues are then given by

$$\frac{s_{ii}}{t_{ii}}, i = 1, \dots, 2n,$$

where $t_{ii}$ and $s_{ii}$ are the diagonal elements of $T$ and $S$, respectively. For general matrix pairs, it is possible that $t_{ii} = 0$ for some $i$, corresponding to an infinite eigenvalue. However, gyroscopic systems with $M > 0$ have only finite eigenvalues. If a real linearization $(A, B)$ is chosen, then a real generalized Schur form exists, where $Q$ and $Z$ are orthogonal, and $T$ and $S$ are upper quasi-triangular. The method described above is the QZ algorithm, a numerically stable method for determining generalized eigenvalues (see [13]).

The QZ algorithm does not take advantage of the structure of the system matrices or the special spectral properties of gyroscopic systems. If $(A, B)$ is chosen to be one of the skew-Hamiltonian/Hamiltonian or Hamiltonian/skew-Hamiltonian linearizations given in (9) and (10), the problem can be reformulated as a Hamiltonian eigenvalue problem. Such problems can be solved using specialized algorithms, such as the one described in [2].

A concise overview of numerical methods for the generalized eigenvalue problem, along with their properties, is provided in [19].

Another numerical challenge in stability analysis is determining whether the eigenvalues are semisimple. Simple eigenvalues are always semisimple, but for multiple eigenvalues (those with an algebraic multiplicity greater than 1), it is necessary to verify their geometric multiplicity. For a generalized eigenvalue $\lambda$ of the matrix pair $(A, B)$, the geometric multiplicity is defined as the dimension of the null space of $A - \lambda B$. To compute the geometric multiplicity, we can calculate the rank of $A - \lambda B$. This method is particularly effective for small to medium-sized matrices and when the eigenvalues are well-separated.

## 5. Numerical examples

In this section, we demonstrate the advantages of using a numerical approach for stability analysis of gyroscopic systems compared to algebraic necessary/sufficient conditions. In the following example, we demonstrate how the numerical approach can provide a significantly larger stability region in a parameter-dependent problem.

**Example 1.** *Let's consider a widely used example of a gyroscopic system commonly found in literature (e.g., see [4, 9, 11]). This example represents a simplified model of a disk mounted on a weightless, non-circular rotating shaft subjected to a constant axial compression force. The system can be described by the following differential equation:*

$$M\ddot{x}(t) + G\dot{x}(t) + Kx(t) = 0,$$

*where*

$$M = I, \quad G = 2\omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} k_1 - \eta - \omega^2 & 0 \\ 0 & k_2 - \eta - \omega^2 \end{bmatrix}. \tag{18}$$

*The parameter $\omega$ represents the shaft's angular velocity, and its value in this example is $\omega = 2$. Furthermore, $\eta$ represents the axial force, with a value of $\eta = 3$. The parameters $0 < k_1 < 7$ and $0 < k_2 < 7$ denote the elastic rigidities in the two principal directions.*

*The paper [4] analyzes what stability area various analytical criteria give for this example. In that paper, the best result gives Bulatovic's criterion, which is applicable only to gyroscopic systems with negative definite $K$:*

**Theorem 5.1.** *(see [4]) If $G^T = -G$ and $K^T = K < 0$, the system $\ddot{x} + G\dot{x} + Kx = 0$ is stable if*

$$4K - G^2 - \delta I > 0 \tag{19}$$

*where $\delta = 2(\delta_1 - \delta_2)$, where $\delta_1$ and $\delta_2$ are the maximal and the minimal eigenvalues of $-K$, respectively.*

*Previous criterion gives that the system is stable if (see [4])*

$$2(k_1 - 3) - |k_1 - k_2| > 0 \quad \text{and} \quad 2(k_2 - 3) - |k_1 - k_2| > 0,$$

*which reduces to*

$$k_1 > 3 \quad \text{and} \quad (6 + k_1)/3 < k_2 < -3 + 2k_1,$$

*and stability area is illustrated in Figure 1(a).*

*On the other hand, the following Barkwell and Lancaster (see [1]) stability criterion gives better results for this example.*

**Theorem 5.2.** *[1, Theorem 5] If $G$ is a real skew-symmetric nonsingular matrix, and $K$ is a real negative definite matrix, then the matrix polynomial $\mathcal{H}(\lambda) = \lambda^2 I + \lambda iG - K$ has all real and semisimple eigenvalues whenever*

$$|G| \geq kI - k^{-1}K \tag{20}$$

*holds for some $k > 0$, where $|G|$ denotes the positive square root of $-G^2$.*

*For this particular example, criterion '$|G| > kI - k^{-1}K$, holds for some $k > 0$' means that*

$$k_1, k_2 > k^2 - 4k + 7 \text{ for some } k > 0. \tag{21}$$

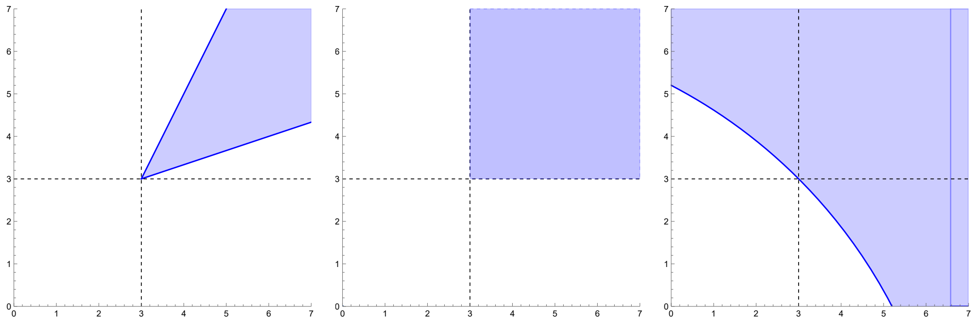


Figure 1: (a) Stability area from [4] (b) Stability area from [1] (c) Numerical stability area

*Since*

$$\min_{k>0}(k^2 - 4k + 7) = 3,$$

*this stability criterion gives a wider stability area, so criterion* (20) *is better then* (19) *for the observed example (see Figure 1(b)). Using our direct approach, it is easy and very straightforward (it can also be calculated by hand because of such small system matrices) to see that some parts of the stability area are not included in the previous two results. An example of this was obtained when we calculated the eigenvalues of the system for* $k_1 = 2.3$ *and* $k_2 = 3.65$ *(which is not included in any stability region defined by previous results). The eigenvalues are* $\pm 2.052i$ *and* $\pm 1.9337i$*, which are purely imaginary and semisimple, implying stability. Similarly, the numerical method gives many more configurations that ensure the stability of the system, and the whole area is illustrated in Figure 1(c). We can see that the stability area is much wider than the analytical criteria can confirm.*

**Example 2.** *In this example, we examine a system similar to the one in the previous example, but with two discs. Specifically, we consider a simplified model consisting of two discs mounted on a non-circular, weightless, elastic shaft that rotates at a constant angular velocity* $\omega$*. An example of the system matrices for this type of system is*

$$M = I, \quad G = 2\omega \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 2-\omega^2 & 1 & 0 & 0 \\ 1 & 4-\omega^2 & 0 & 0 \\ 0 & 0 & 2-\omega^2 & 1 \\ 0 & 0 & 1 & 4-\omega^2 \end{bmatrix}.$$

*We observe the stability of this system with respect to* $\omega \geq 0$*. This results in* $K$*, which can be positive definite (e.g., for* $\omega = 1$*), indefinite (e.g., for* $\omega = 2$*), or negative definite (e.g., for* $\omega = 3$*). This means that none of the analytical criteria that rely on the assumption of the definiteness of* $K$ *can be applied to all three cases (such as those given in Theorems 5.1 and 5.2 from the previous example). If we notice that* $K$ *and* $G$ *commute, we can apply the analytical criterion* (5) *and verify the positive definiteness of* $4K - G^2$ *to conclude that the system is stable in all three cases. Indeed, numerical methods also confirm stability (the obtained eigenvalues are given in Table 1), but without requiring verification of any conditions on the system matrices, which gives them an advantage in practical applications.*

**Example 3.** *In this example, we analyze the stability of a gyroscopic system of dimension* $n = 10$*, where the mass matrix is a random diagonal matrix with diagonal elements in* $[0, 1]$*, the gyroscopic matrix is a full random antisymmetric matrix with elements from* $[-100, 100]$*, and the stiffness matrix is a random symmetric matrix with eigenvalues in the range* $[-100, 1]$

| $\omega$ | eigenvalues |
|---|---|
| 1 | $\pm 3.1010\,i$, $\pm 2.2593\,i$, $\pm 0.2593\,i$, $\pm 1.1010\,i$ |
| 2 | $\pm 4.1010\,i$, $\pm 3.2593\,i$, $\pm 0.7407\,i$, $\pm 0.1010\,i$ |
| 3 | $\pm 5.1010\,i$, $\pm 4.2593\,i$, $\pm 1.7407\,i$, $\pm 0.8990\,i$ |

Table 1: Eigenvalues of QEP from Example 2 for $\omega \in \{1, 2, 3\}$

*(hence, $K$ can be negative (semi)definite or indefinite). We repeated the experiment 100 times. In six instances, the generated system had an indefinite $K$, and all corresponding systems were unstable. In the remaining* 94 *cases, the generated system had a negative definite $K$ and was stable in* 19 *instances. This analysis does not rely on any prior assumptions about the interconnections between the system matrices, and the most effective approach to analyze it is through numerical procedures.*

## 6. Conclusions

In this paper, the stability problem of gyroscopic systems is considered. Stability is related to the eigenvalues of the associated quadratic problem and their multiplicities. Many papers provide analytical conditions for stability verification, which are simpler than computing generalized eigenvalues, but their application is very restrictive, so they often do not give the results. On the other hand, numerical methods always provide results, and when stable algorithms are used, obtained results are reliable. In the examples in Chapter 5, it can be seen how the numerical criterion provides a significant advantage compared to some algebraic methods in checking the stability or in determining the stability region for a specific parameter-dependent problems. These results indicate that numerical methods should be used in practice, especially when analytical criteria are not applicable.

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