An input relaxation model for evaluating congestion in fuzzy DEA

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Abstract. This paper develops a BCC input relaxation model for identifying input congestion as a severe form of inefficiency of decision-making units in fuzzy data envelopment analysis. The possibility approach is presented to obtain the models equivalent to fuzzy models. We use a one-model approach to determine input congestion based on the BCC input relaxation model. A numerical example is given to illustrate the proposed model and identify the congestion with precise and imprecise data. The proposed model is also used to determine the congestion in 16 hospitals using four fuzzy inputs and two fuzzy outputs with a symmetrical triangular membership function.

1. Introduction

Data envelopment analysis (DEA) is a mathematical programming technique for evaluating the relative performance of decision-making units (DMUs), widely used in almost every field, such as agriculture, banking, benchmarking, economy, edu-

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cation, environment, government, health, human resources, information technology, insurance, manufacturing, marketing, operations, public policy, regulation, retail, services, and tourism, having emerged as a powerful management science tool [3]. DEA was introduced by Charnes, Cooper, and Rhodes [4], and the first model in DEA was called the CCR model. Banker, Charnes, and Cooper [1] added the convex constraint to the CCR model and developed a new model that was called the BCC model.

Dealing with uncertainty is unavoidable in modelling and measuring performance. Incorporating uncertainty in DEA modelling has been introduced by many researchers. Recent details can be found in Charles and Kumar [2], Shiraz, Charles, and Jalalzadeh [30], and Tsolas and Charles [32].

One of the recent interests is focusing on discussing congestions in DEA modelling, under uncertain environments. Input congestion, as a severe form of technical inefficiency, has been investigated by many researchers and different models have been employed in DEA to identify the source of the input congestion and estimate its values with crisp and imprecise data. For example, Cooper *et al.* [9] proposed a one-model approach for determining technical inefficiency and congestion in the inputs of DMUs with associated measures in DEA. Cooper *et al.* [11] extended the ordinary DEA models for identifying and determining the input congestion to chance-constrained programming approaches with stochastic inputs and outputs in DEA. Kheirollahi *et al.* [19] used a BCC input relaxation model to identify the input congestion of Iranian hospitals using stochastic DEA with chance-constrained programming approaches. Flegg and Allen [15] applied DEA models to analyze and determine the input congestion in new British universities. Noura *et al.* [27] introduced a new method, called Noura's approach, to identify and determine the amount of the input congestion of DMUs in DEA. Noura's approach requires considerably fewer computations when compared to the other models. Färe and Grosskopf [14] focused on clarifying the distinction between the approaches to modeling technology and measuring congestion, differentiating between weak disposability and the law of variable proportions. Finally, Saljooghi and Rayeni [29] investigated a methodology to measure the undesirable output congestion, as well as the input congestion, from both theoretical and empirical perspectives. To this end, they applied their proposed approach to identify congestion and measure the undesirable output in the Sistan and Baluchistan University's educational departments in Iran.

Traditionally, DEA models require accurate and crisp data, since DEA is a methodology focused on the frontier of production possibility set (PPS). A few changes in the data can change the efficient frontier significantly. Therefore, some authors proposed fuzzy approaches to assess the efficiency of homogeneous DMUs, determining the congestion and estimating the returns to scale. For instance, Khodabakhshi, Gholami, and Kheirollahi [22] used an additive model to provide an alternative approach for estimating the returns to scale in both stochastic DEA

and fuzzy DEA. Lertworasirikul *et al.* [25] introduced the possibility, credibility, and necessity approaches to solve fuzzy DEA models. They explained that by using the possibility, credibility, and necessity approaches, the relationship between the primal and dual models of the fuzzy BCC models is revealed and fuzzy efficiency can be constructed. Khodabakhshi and Hejrizadeh [23] developed the fuzzy version of the input relaxation model by using some ranking methods based on the comparison of α-cuts. Rostamy-Malkhalifeh *et al.* [28] proposed a new approach for determining the input congestion of DMUs with fuzzy inputs and outputs. Hosseinzadeh Lotfi *et al.* [16] used the one-model approach to estimate the input congestion in five Iranian banks, with two stochastic inputs and two outputs. A review of the different approaches to identifying the input congestion of DMUs in DEA can be found in Khodabakhshi, Hosseinzadeh Lotfia, and Aryavash [24].

The aim of the present paper is to determine the input congestion of DMUs using a BCC input relaxation model with precise and imprecise data. The possibility approach is used to convert the nonlinear model to a linear model. We assume that there are n DMUs, DMUj, $j = 1, 2, ..., n$, wherein each of them produces s nonnegative different outputs $y_{rj} (r = 1, 2, ..., s)$, using m nonnegative different inputs $x_{ij} (I = 1, 2, ..., m)$. Sometimes, in evaluating the performance of the DMUs, we need to decrease some inputs and increase other inputs to increase outputs. To this end, the BCC input relaxation model is applied as follows [17, 21]:

$$
\begin{aligned}
& Maximize \quad \phi + \varepsilon (\sum_{i=1}^{m} s_{i1}^{-} - \sum_{i=1}^{m} s_{i2}^{+} + \sum_{r=1}^{s} s_r^{+}) \\
& \text{Subject to} \quad \sum_{j=1}^{n} \lambda_j y_{rj} - y_{ro} - s_r^{+} = 0, \qquad r = 1, 2, ..., s, \\
& \qquad \sum_{j=1}^{n} \lambda_j x_{ij} + s_{i1}^{-} - s_{i2}^{+} = x_{io}, \qquad i = 1, 2, ..., m, \qquad (1) \\
& \qquad \qquad \sum_{j=1}^{n} \lambda_j = 1, \\
& s_{i1}^{-}, s_{i2}^{+}, s_r^{+}, \lambda_j \geq 0, \; i = 1, 2, ..., m; \; r = 1, 2, ..., s; \; j = 1, 2, ..., n.
\end{aligned}
$$

In the above model, the symbol $j = 0$ designates one of the DMUj as the DMUo to be evaluated relative to all the data (including the data on DMUj = DMUo), where $\phi \geq 1$ is a real number. The term ε in the objective function of model (1) is a non-Archimedean, which is smaller than any positive real number, and which is used in order to optimize the slacks and ensure that all of the slacks are

considered in the optimal solution. The variables s_{i1}^{-} and s_{i2}^{+} are the slacks for the decrement and increment of the ith input, respectively. Finally, s_r^{+} is a slack for the increment of the rth output.

Definition 1 (Efficiency): DMUo is efficient under model (1) if the following conditions are satisfied:

1) $\phi^{*} = 1$;

2) $s_{i1}^{-*} = s_{i2}^{+*} = s_r^{+*} = 0$; for $i = 1, 2, ..., m$ and $r = 1, 2, ..., s$.

Here and hereafter * is used to designate an optimal value.

The rest of this article is organized as follows: In Section 2, the definition of congestion and the one-model approach based on the relaxed combination of inputs for determining congestion are provided, along with a numerical example in the subsequent Section 3. Section 4 develops the one-model approach to the BCC input relaxation model with the triangular fuzzy data. A fuzzy numerical example is also presented in Section 5 to evaluate the efficiency and determine the congestion via a fuzzy BCC input relaxation model. The proposed model is then applied to identify the input congestion of 16 hospitals by using fuzzy inputs and outputs with a symmetrical triangular membership function. Conclusions are provided in Section 7.

2. Congestion

Congestion, as the severe form of inefficiency, occurs when the excessive amounts of input cause a reduction in the output of DMUs. Researchers have investigated the input congestion of DMUs with precise and imprecise inputs and outputs, in the literature. Among the advantages of identifying and managing congestion, we can mention the increase in output, as well as the elimination of the increase in the input of DMUo. In DEA, many models have been introduced to identify input congestion of DMUs. The most famous model, which is called the one-model approach, was introduced by Cooper *et al.* [11]. This model was then expanded into the BCC input relaxation model.

In this paper, the one-model approach to the BCC input relaxation model is applied to identify the input congestion with fuzzy inputs and outputs and the possibility approach is used to obtain the deterministic models equivalent to the fuzzy models. Cooper *et al.* [10] defined congestion as follows:

Definition 2 (Congestion): Congestion is present in the performance of DMUo, the DMU being evaluated, when input increases give rise to decreases in the outputs that are maximally attainable or, conversely, when input decreases are accompanied by increases in the output amounts that are maximally attainable.

We propose a BCC input relaxation model for identifying the congestion of the DMUs, following Cooper *et al.* [10]:

$$Maximize \quad \phi + \varepsilon(\sum_{r=1}^{s} s_r^+ - \sum_{i=1}^{m} s_i^{-c} - \sum_{i=1}^{m} s_{i2}^+)$$

$$\text{Subject to} \quad \sum_{j=1}^{n} \lambda_j y_{rj} - y_{ro} - s_r^+ = 0, \quad r = 1, 2, ..., s,$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^{-c} - s_{i2}^+ = x_{io}, \quad i = 1, 2, ..., m, \tag{2}$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$s_i^{-c}, s_{i2}^+, s_r^+, \lambda_j \geq 0, \ i = 1, 2, ..., m; \ r = 1, 2, ..., s; \ j = 1, 2, ..., n.$$

The congestion amount of the input of DMUj, $j = 1, 2, ..., n$, is represented by s_i^{-c*} in model (2), in accordance with the following theorem [20].

Theorem 1: Congestion is present if and only if in the optimal solution $(\phi^*, \lambda^*, s_i^{-c*}, s_{i2}^{+*}, s_r^{+*})$ of model (2), at least one of the following conditions is satisfied:

1) $\phi^* > 1$ and there is at least one $s_i^{-c*} > 0$, for $i = 1, 2, ..., m$;

2) There exists at least one $s_r^{+*} > 0$, for $r = 1, 2, ..., s$, also one $s_i^{-c*} > 0$, for $i = 1, 2, ..., m$.

Both the conditions of Theorem 1 tell us that DMUo is inefficient. That is, if a DMU has input congestion, then it is inefficient, but the reverse is not true.
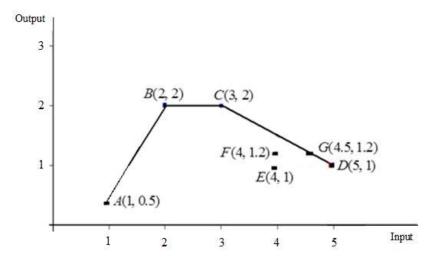
Figure 1: *Input congestion*

3. Numerical example

Now, we use the BCC input relaxation model (2) to identify the input congestion of the DMUs A, B, C, D, E, F, and G, as depicted in Figure 1. These DMUs apply one input to produce one output. For point A in Figure 1, we apply model (2) to obtain the following problem:

$$Maximize \quad \phi + \varepsilon(s^+ - s^{-c} - s_{12}^+)$$

$$Subject to \quad 0.5\lambda_A + 2\lambda_B + 2\lambda_C + \lambda_D + \lambda_E + 1.2\lambda_F + 1.2\lambda_G - s^+ = 0.5\phi$$

$$1\lambda_A + 2\lambda_B + 3\lambda_C + 5\lambda_D + 4\lambda_E + 4\lambda_F + 4.5\lambda_G + s^{-c} - s_{12}^+ = 1 \quad (3)$$

$$\lambda_A + \lambda_B + \lambda_C + \lambda_D + \lambda_E + \lambda_F + \lambda_G = 1$$

$$\lambda_A, \lambda_B, \lambda_C, \lambda_D, \lambda_E, \lambda_F, \lambda_G, s^{-c}, s_{12}^+, s^+ \geq 0$$

The optimal solution of the above problem is $\lambda_C^* = 1$, $\phi^* = 4$, $s^{+*} = 0$, $s_{12}^{+*} = 2$, $s^{-c*} = 0$ and the other variables are zero. Based on part 1 of Theorem 1, since $\phi^* = 4 > 1$, DMU A is inefficient and since $s^{-c*} = 0$, this DMU doesn't have congestion in its input. In addition, we have $s_{12}^{+*} = 2$; therefore, this DMU can increase its input by two units without worsening its efficiency. For DMU B, the

right-hand side of the first and second constraints in problem (3) is modified to 2 and 2ϕ, respectively. In this case, the optimal solution is $\lambda_C^* = 1$, $\phi^* = 1$, $s^{+*} = 0$, $s_{12}^{+*} = 1$, and $s^{-c*} = 0$. This optimal solution shows that Theorem 1 is not satisfied and there is no congestion present in DMU B. Also, since $s_{12}^{+*} = 1$, DMU B can increase its input by one unit without worsening its efficiency. For DMU C, the optimal solution of the modified problem is $\lambda_C^* = 1$, $\phi^* = 1$, $s^{+*} = 0$, $s_{12}^{+*} = 0$, and $s^{-c*} = 0$. None of the conditions of Theorem 1 is satisfied. Thus, this DMU is efficient and there is no congestion present in its input. The optimal solution of the modified problem for DMU D is $\lambda_C^* = 1, \phi^* = 2$, $s^{+*} = 0$, $s_{12}^{+*} = 0$, and $s^{-c*} = 2$. Since $\phi^* = 2$ and $s^{-c*} = 2$, part 1 of Theorem 1 is satisfied and congestion is present in DMU D and the congestion amount in its input is $s^{-c*} = 2$. Finally, the results of using model (2) for all DMUs in Figure 1, are presented in Table 1.

DMUs	ϕ^*	s^{-c*}	s_{12}^{+*}	s^{+*}
A	4.000	0	1	0
B	1.000	0	1	0
C	1.000	0	0	0
D	2.000	2	0	0
E	1.000	1	0	0
F	1.667	1	0	0
G	1.667	1.5	0	0

Table 1: *Determining congestion by the BCC input relaxation model (2)*

4. Input relaxation congestion model in fuzzy DEA

Uncertainty is inevitable and commonly exists in many decision-making problems. For this reason, the ordinary set theory has been developed into the fuzzy set theory, in which the problem of uncertainty is studied. The fuzzy set theory was first introduced by Zadeh [34], suggesting that fuzzy sets can be used as a basis for the theory of possibility similar to the way that the measure theory provides a basis for the theory of probability. More information about the possibility theory can be found in Dubois and Prade [12].
Zadeh [34] introduced a fuzzy variable, which is associated with a possibility distribution in the same manner that a random variable is associated with a

probability distribution. Using the possibility theory, the possibilities of the fuzzy events (i.e., fuzzy constraints) can be determined. In this section, following Jahanshahloo and Khodabakhshi [17], a BCC input relaxation model for improving the output with fuzzy input and output data (FIR) is given as follows:

$$\textit{Maximize} \qquad \phi$$

$$\text{Subject to} \qquad \sum_{j=1}^{n} \lambda_j \widetilde{y}_{rj} \geq \phi \widetilde{y}_{ro}, \quad r = 1, 2, \ldots, s,$$

$$\sum_{j=1}^{n} \lambda_j \widetilde{x}_{tj} - s_{i2}^{+} \leq \widetilde{x}_{io}, \quad i = 1, 2, \ldots, m, \qquad (4)$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$s_{i2}^{+}, \lambda_j \geq 0, \quad i = 1, 2, \ldots, m; \, j = 1, 2, \ldots, n.$$

The concept of chance-constrained programming (CCP), which was introduced by Charnes and Cooper [5], is adopted to solve the FIR model (4). CCP deals with uncertainty by specifying the desired levels of confidence for which the constraints hold. Using the concepts of the CCP and the possibility of the fuzzy events or fuzzy constraints, the FIR model (4) becomes the possibility input relaxation (PIR) model (5):

$$\textit{Maximize} \qquad \phi$$

$$\text{Subject to} \qquad \pi(\sum_{j=1}^{n} \lambda_j \widetilde{y}_{rj} \geq \phi \widetilde{y}_{ro}) \geq \beta_r, \quad r = 1, 2, \ldots, s,$$

$$\pi(\sum_{j=1}^{n} \lambda_j \widetilde{x}_{ij} - s_{i2}^{+} \leq \widetilde{x}_{io}) \geq \alpha_i \quad i = 1, 2, \ldots, m, \qquad (5)$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$s_{i2}^{+}, \lambda_j \geq 0, \quad i = 1, 2, \ldots, m; \, j = 1, 2, \ldots, n.$$

where $\beta_r \in [0,1]$, $r = 1, 2, \ldots, s$, and $\alpha_i \in [0,1]$, $i = 1, 2, \ldots, m$, are prespecified acceptable levels of possibility and π is the possibility measure from $P(\Theta_i)$ to $[0, 1]$ in which Θ_i is an arbitrary set and $P(\Theta_i)$ is the collection of all subsets of Θ_i. Let $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_s]$ and let us define an α- possibilistic efficient DMU and an α - possibilistic inefficient DMU, as follows:

Definition 3 (Fuzzy efficiency): DMUo with fuzzy inputs and outputs is α-possibilistic efficient if and only if the following two conditions hold:

1) $\phi^* = 1$,

2) The slack values are all zeros for all optimal solutions.

However, to make a reasonable efficiency comparison of the DMUs, the possibility levels of the constraints α in the PIR model (5) should be set at the same level. The following lemma is useful in finding the optimal solution of the possibilistic BCC (PBCC) model (5).

Lemma 1 (Lertworasirikul *et al.* [25])**:** Let $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n$ be fuzzy variables with normal and convex membership functions. Suppose $(.)_{\alpha_j}^L$ and $(.)_{\alpha_j}^U$ denote the lower and upper bounds of the α-level set of $\tilde{a}_i$, $(i = 1, 2, ..., m)$. Then, for any given possibility levels α_1, α_2 and α_3 with $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1$:

1. $\pi(\tilde{a}_1 + \tilde{a}_2 + ... + \tilde{a}_n \leq b) \geq \alpha_1$ if and only if
$$(\tilde{a}_1)_{\alpha_1}^L + (\tilde{a}_2)_{\alpha_1}^L + ... + (\tilde{a}_n)_{\alpha_1}^L \leq b,$$

2. $\pi(\tilde{a}_1 + \tilde{a}_2 + ... + \tilde{a}_n \geq b) \geq \alpha_2$ if and only if
$$(\tilde{a}_1)_{\alpha_2}^U + (\tilde{a}_2)_{\alpha_2}^U + ... + (\tilde{a}_n)_{\alpha_2}^U \geq b,$$

3. $\pi(\tilde{a}_1 + \tilde{a}_2 + ... + \tilde{a}_n = b) \geq \alpha_3$ if and only if
$$(\tilde{a}_1)_{\alpha_3}^L + (\tilde{a}_2)_{\alpha_3}^L + ... + (\tilde{a}_n)_{\alpha_3}^L \leq b \quad \text{and} \quad (\tilde{a}_1)_{\alpha_3}^U + (\tilde{a}_2)_{\alpha_3}^U + ... + (\tilde{a}_n)_{\alpha_3}^U \geq b.$$

The PBCC model (5) can be transformed into:

$$\textit{Maximize} \qquad \phi$$

$$\text{Subject to} \qquad \pi(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \phi \tilde{y}_{ro} \geq 0) \geq \beta_r, \quad r = 1, 2, ..., s,$$

$$\pi(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} - \tilde{x}_{io} \leq s_{i2}^+) \geq \alpha_i \quad i = 1, 2, ..., m, \qquad (6)$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$s_{i2}^+, \lambda_j \geq 0, \quad i = 1, 2, ..., m; j = 1, 2, ..., n.$$

Depending upon the membership functions of the fuzzy parameters in model (6), the model may take the form of a linear programming or a nonlinear one. Based

on lemma 1 and following Liu [26], for a trapezoidal fuzzy number $[(r_i)^L_\circ, (r_i)^L_1, (r_i)^U_1, (r_i)^U_\circ]$, $i = 1,2,...,m$, and any given possibility level α, $0 \le \alpha \le 1$ the following relations hold:

1) $\pi(\tilde{r}_1 + \tilde{r}_2 + ... + \tilde{r}_n \le b) \ge \alpha$ if and only if

$$(1-\alpha)[(\tilde{r}_1)^L_\circ + (\tilde{r}_2)^L_\circ + ... + (\tilde{r}_n)^L_\circ] + \alpha[(\tilde{r}_1)^L_1 + (\tilde{r}_2)^L_1 ... + (\tilde{r}_n)^L_1] \le b$$

2) $\pi(\tilde{r}_1 + \tilde{r}_2 + ... + \tilde{r}_n \ge b) \ge \alpha$ if and only if

$$(1-\alpha)[(\tilde{r}_1)^U_\circ + (\tilde{r}_2)^U_\circ + ... + (\tilde{r}_n)^U_\circ] + \alpha[(\tilde{r}_1)^U_1 + (\tilde{r}_2)^U_1 + ... + (\tilde{r}_n)^U_1] \ge b$$

3) $\pi(\tilde{r}_1 + \tilde{r}_2 + ... + \tilde{r}_n = b) \ge \alpha$ if and only if

$$(1-\alpha)[(\tilde{r}_1)^L_\circ + (\tilde{r}_2)^L_\circ + ... + (\tilde{r}_n)^L_\circ] + \alpha[(\tilde{r}_1)^L_1 + (\tilde{r}_2)^L_1 + ... + (\tilde{r}_n)^L_1] \le b$$

$$(1-\alpha)[(\tilde{r}_1)^U_\circ + (\tilde{r}_2)^U_\circ + ... + (\tilde{r}_n)^U_\circ] + \alpha[(\tilde{r}_1)^U_1 + (\tilde{r}_2)^U_1 + ... + (\tilde{r}_n)^U_1] \ge b .$$

Therefore, when inputs and outputs are trapezoidal fuzzy numbers, by adding the slack variables, model (6) becomes a linear programming model, as follows:

$$\textit{Maximize} \qquad \phi + \varepsilon(\sum_{i=1}^{m} s^-_{i1} - \sum_{i=1}^{m} s^+_{i2} + \sum_{r=1}^{s} s^+_r)$$

$$\text{Subject to} \quad (1-\beta_r)(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \phi\tilde{y}_{ro})^U_0 + \beta_r(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \phi\tilde{y}_{ro})^U_1 - s^+_r = 0,$$

$$r = 1, 2, ..., s, \qquad\qquad (7)$$

$$(1-\alpha_i)(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} - \tilde{x}_{io})^L_0 + \alpha_i(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} - \tilde{x}_{io})^L_1 + s^-_{i1} - s^+_{i2} = 0, \ i = 1, 2, ..., m,$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$s^-_{i1}, s^+_{i2}, s^+_r, \lambda_j \ge 0, \ i = 1, 2, ..., m; \ r = 1, 2, ..., s; \ j = 1, 2, ..., n.$$

Model (7) is a generalization of the BCC model (1) with fuzzy inputs and outputs. In a similar way, one can generalize the one-model approach to congestion presented in model (2) to the following version:

$$\textit{Maximize} \qquad \phi + \varepsilon(\sum_{r=1}^{s} s^+_r - \sum_{i=1}^{m} s^{-c}_i - \sum_{i=1}^{m} s^+_{i2})$$

$$\text{Subject to} \quad (1-\beta_r)(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \phi\tilde{y}_{ro})^U_0 + \beta_r(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \phi\tilde{y}_{ro})^U_1 - s^+_r = 0,$$

$$r = 1, 2, ..., s,$$

$$(1-\alpha_i)\,(\sum_{j=1}^{n}\lambda_j\tilde{x}_{ij}-\tilde{x}_{io})_0^L+\alpha_i(\sum_{j=1}^{n}\lambda_j\tilde{x}_{ij}-\tilde{x}_{io})_1^L+s_i^{-c}-s_{i2}^+=0,$$

$$i=1,\,2,\,...,\,m, \qquad\qquad (8)$$

$$\sum_{j=1}^{n}\lambda_j=1,$$

$$s_i^{-c},s_{i2}^+,s_r^+,\lambda_j\geq0,\;\;i=1,\,2,\,...,\,m;\;r=1,\,2,\,...,\,s;\;j=1,\,2,\,...,\,n.$$

Therefore, we have the following theorem as an obvious generalization of Theorem 1:

Theorem 2: Suppose $(\phi^*,\lambda^*,s_i^{-c*},s_{i2}^{+*}\,s_r^{+*})$ is the optimal solution of model (8). Then, at the possibility level α, DMUo has input congestion if and only if at least one of the following conditions is satisfied:

1) $\phi^*>1$, and there exists at least one $s_i^{-c*}>0$, for $i=1,\,2,\,...,\,m$.

2) There exists at least one $s_r^{+*}>0$, for $r=1,\,2,\,...,\,s$, also one $s_i^{-c*}>0$, for $i=1,\,2,\,...,\,m$.

5. Fuzzy numerical example

We present a numerical experiment to apply the possibility approach to identify the input congestion via the BCC input relaxation model (8), when inputs and outputs are fuzzy numbers. The example is taken from Guo and Tanaka [18]. The fuzzy inputs and fuzzy outputs are given in Table 2. These fuzzy inputs and outputs have symmetrical triangular membership functions, a special case of the trapezoidal membership functions. The membership functions are denoted by (c, d) where c is the center and d is the spread. In this example, all the fuzzy constraints are satisfied with the same possibility level, *i.e*,

$$\beta_1=\beta_2=...=\beta_s=\alpha_1=\alpha_2=...=\alpha_m.$$

DMU	DMU1	DMU2	DMU3	DMU4	DMU5
Input 1	(4,0.5)	(2.9,0)	(6.9,0.5)	(4.1,0.7)	(8.5,0.6)
Input 2	(5.1,0.2)	(1.5,0.1)	(8.6,0.4)	(2.3,0.4)	(7.1,0.5)
Output 1	(4.6,0.2)	(5.2,0)	(3.2,0.3)	(2.9,4)	(5.1,0.7)
Output 2	(4.1,0.3)	(4.5,0.2)	(5.1,0.8)	(5.7,0.2)	(7.4,0.9)

Table 2: *DMUs with two fuzzy inputs and two fuzzy outputs*

The results for the possibility levels 0, 0.25, 0.5, 0.75, and 1 are provided in Table 3. In this Table, the numbers in parentheses represent the efficiency value, the inputs congestions of the corresponding $(\phi^*, s_1^{-c*}, s_2^{-c*})$, evaluated at the given possibility level α. The interpretation of the results is as follows. At the possibility level 1, DMU2 and DMU5 are possibilistically efficient and DMU1, DMU3, and DMU4 are possibilistically inefficient, and their congestion values corresponding to the first input are 0.26, 0.455, and 0.43, respectively. Note that in the second input, there is no congestion for these DMUs. From Table 3, DMU2 and DMU5 are possibilistically efficient at all the possibility levels, whereas DMU3 and DMU4 are possibilistically inefficient at some possibility levels. In addition, for DMU3 and DMU4 there is congestion at some possibility levels, and DMU1 is inefficient at all the possibility levels.

α	DMU1	DMU2	DMU3	DMU4	DMU5
0	(1.072,0,0)	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
0.25	(1.079,0.226,0)	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
0.5	(1.088,0,0)	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
0.75	(1.092,0.18,0)	(1,0,0)	(1.057,0.494,0)	(1,0,0)	(1,0,0)
1	(1.096,0.260,0)	(1,0,0)	(1.057,0.455, 0)	(1.1,0.43,0)	(1,0,0)

Table 3: *The results of the efficiency values and congestion at 5 possibility levels*

6. Empirical example

Now, we apply the fuzzy congestion model (8) to identify the input congestion; to this end, we employ the technical data for 16 hospitals, which are presented in Table 4. These fuzzy inputs and fuzzy outputs have symmetrical triangular membership functions. The data set is taken from Tone and Sahoo [31]. The inputs are the total hours worked by doctors in the survey period (I1), total hours worked by nurses (I2), total hours worked by technical workers (I3), and total hours worked by office staff (I4). The outputs are the total medical insurance points for outpatients (O1) and the total medical insurance points for inpatients (O2).

DM-Us	Hospitals	I1	I2	I3	I4	O1	O2
1	H1	(995, 17)	(6205, 262)	(1375, 35)	(2629, 49)	(4127, 83)	(1678, 17)
2	H2	(917,15)	(5898,248)	(1379,37)	(2047,37)	(3721,77)	(1277,13)
3	H3	(3178,91)	(10049,349)	(3615,85)	(3511,61)	(2706,27)	(2051,20)
4	H4	(813,14)	(5833,233)	(1124,34)	(1730,27)	(2176,21)	(1538,15)
5	H5	(1236,21)	(8639,309)	(2486,56)	(4990,92)	(5220,105)	(20426,204)
6	H6	(1146,11)	(7610,289)	(1600,30)	(3589,59)	(3517,65)	(1856,19)
7	H7	(705,13)	(5600,200)	(1557,27)	(3623,63)	(2352,24)	(20606,206)
8	H8	(2871,89)	(11524,404)	(2880,51)	(2452,52)	(1755,18)	(1664,17)
9	H9	(1089,20)	(8998,228)	(1730,40)	(2823,53)	(4412,89)	(2334,24)
10	H10	(2032,52)	(9383,243)	(2421,47)	(4454,84)	(5386,111)	(2080,21)
11	H11	(1414,24)	(10468,263)	(2140,57)	(3649,69)	(5735,118)	(2691,27)
12	H12	(1967,29)	(11260,272)	(2759,69)	(3178,48)	(6079,125)	(2804,29)
13	H13	(1851,28)	(9880,250)	(2335,55)	(4570,90)	(5893,113)	(2495,26)
14	H14	(3100,102)	(15649,479)	(5487,127)	(2940,55)	(5248,103)	(3692,37)
15	H15	(5016,176)	(18010,480)	(4008,98)	(3567,60)	(7800,131)	(4582,47)
16	H16	(1924,35)	(12682,452)	(2490,63)	(2975,53)	(6040,123)	(3396,35)

Note: I1, I2, I3, and I4 denote the inputs; O1 and O2 denote the outputs.

Table 4: *Imprecise data for four inputs and two outputs, for 16 hospitals*

The computational results of the BCC input relaxation model (8) for determining the input congestion of hospitals are shown in Table 5. The first column of Table 5 shows the hospital's number and columns 2, 3, and 4 show the computational results of model (8) at the possibility levels of 0, 0.5, and 1, respectively. In Table 5, the numbers in parentheses are the efficiency score, ϕ^*, the congestion of the third and fourth input (s_3^{-c*} and s_4^{-c*}), and the slacks of the outputs (s_1^{+*} and s_2^{+*}) in evaluating the hospitals. The congestion for the first and second inputs for all of the hospitals has not been shown, due to their absence. Based on the results provided in Table 5, hospitals 5, 7, and 15 have the optimal solutions $\phi^* = 1$, $s_3^{-c*} = 0$, $s_4^{-c*} = 0$, $s_1^{+*} = 0$, and $s_2^{+*} = 0$; so these hospitals don't have input congestion. The other hospitals with the efficiency score of $\phi^* > 1$ are inefficient at all the possibility levels. In fact, in efficient units, the score of efficiency ϕ^*, is equal to one and all the slacks of input and output, including $s^{-c*} = 0$, congestion, are zero.

Hospitals	$(\phi^*, s_3^{-c*}, s_4^{-c*}, s_1^{+*}, s_2^{+*})$ with possibility level α		
	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$
H1	(1.88,0,0,0,1436)	(1.89,0,0,0,1423)	(1.89,0,0,0,1411)
H2	(2.09,0,0,0,1935)	(2.09,0,0,0,1920)	(2.10,0,0,0,1905)
H3	(2.83,0,0,0,0)	(2.82,0,0,0,0)	(2.81,0,0,0,0)
H4	(3.54,0,0,0,0)	(3.53,0,0,0,0)	(3.52,0,0,0,0)
H5	(1,0,0,0,0)	(1,0,0,0,0)	(1,0,0,0,0)
H6	(2.21,0,23,0,478)	(2.22,0,23,0,471)	(2.22,0,22,0,466)
H7	(1,0,0,0,0)	(1,0,0,0,0)	(1,0,0,0,0)
H8	(4.24,0,0,0, 0)	(4.23,0,0,0,0)	(4.22,0,0,0,0)
H9	(1.76,0,0,0,474)	(1.76,0,0,0,466)	(1.77,0,0,0,456)
H10	(1.44,0,863,0,1599)	(1.45,0,875,0,1584)	(1.45,0,887,0,1570)
H11	(1.36,0,73,0,946)	(1.35,0,78,0,934)	(1.36,0,82,0,922)
H12	(1.28,0,0,0,1001)	(1.28,0,0,0,993)	(1.28,0,0,0,984)
H13	(1.32,0,973,0,1300)	(1.32,0,988,0,1290)	(1.32,0,1003,0,1280)
H14	(1.46,1524,0,0,0)	(1.46,1750,0,0,0)	(1.46,1557,0,0,0)
H15	(1,0,0,0,0)	(1,0,0,0,0)	(1,0,0,0,0)
H16	(1.29,0,0,0,214)	(1.29,0,0,0,205)	(1.29,0,0,0,196)

Table 5: *The results of efficiency values and input congestion at 3 possibility levels*

As we can see from Table 5, hospital 6 has an efficiency score of more than one, which means that this hospital is inefficient at all the possibility levels. At the possibility level $\alpha = 0$, the optimal solution of this hospital is $\phi^* = 2.21$, $s_3^{-c*} = 0$, $s_4^{-c*} = 23$, $s_1^{+*} = 0$, and $s_2^{+*} = 478$. This optimal solution satisfies condition (1) of Theorem 2, so in the performance evaluation of hospital 6, the fourth input, namely, the total hours worked by office staff, has congestion, and the amount of its congestion is $s_4^{-c*} = 23$.

The results provided in Table 5 also show that hospital 10 with $\phi^* > 1$ is inefficient at all the possibility levels. At the possibility level zero, the optimal solution of this hospital is $\phi^* = 1.44$, $s_3^{-c*} = 0$, $s_4^{-c*} = 863$, $s_1^{+*} = 0$, and $s_2^{+*} = 1599$. This optimal solution satisfies condition (1) of Theorem 2, so in the performance evaluation of hospital 10, the fourth input has congestion and the amount of its congestion is $s_4^{-c*} = 863$.

Hospital 11 with $\phi^* > 1$ is inefficient at all the possibility levels. At the possibility level zero, the optimal solution of this hospital is $\phi^* = 1.36$, $s_3^{-c*} = 0$, $s_4^{-c*} = 73$

, $s_1^{+*} = 0$, and $s_2^{+*} = 946$. This optimal solution satisfies condition (1) of Theorem 2, so in the performance evaluation of hospital 11, the fourth input has congestion and the amount of its congestion is $s_4^{-c*} = 73$.

Hospital 13 with $\phi^* > 1$ is inefficient at all the possibility levels. At the possibility level zero, the optimal solution of this hospital is $\phi^* = 1.32$, $s_3^{-c*} = 0$, $s_4^{-c*} = 973$, $s_1^{+*} = 0$, and $s_2^{+*} = 1300$. This optimal solution satisfies condition (1) of Theorem 2, so in the performance evaluation of hospital 13, the fourth input has congestion and the amount of its congestion is $s_4^{-c*} = 973$.

Finally, hospital 14 with $\phi^* > 1$ is inefficient at all the possibility levels. At the possibility level zero, the optimal solution of this hospital is $\phi^* = 1.46$, $s_3^{-c*} = 1524$, $s_4^{-c*} = 0$, $s_1^{+*} = 0$, and $s_2^{+*} = 0$. This optimal solution satisfies condition (1) of Theorem 2, so in the performance evaluation of hospital 14, the third input, namely, the total hours worked by technical workers, has congestion and the amount of its congestion is $s_3^{-c*} = 1524$. At the possibility levels of 0, 0.5, and 1, the amount of congestion in the third input of hospital 14 is 1542, 1750, and 1557, respectively.

7. Conclusion

Congestion is a widely observed phenomenon that indicates an economic state where inputs are overly invested [33]. The requirement to study *congestion* is, therefore, rather obvious; furthermore, it has additional important practical implications, especially when its use is associated with a need for increasing inputs to serve vital objectives besides output maximization [10]. As such, in time, congestion has developed into a topic that has received considerable attention from both researchers and practitioners alike. And while there are some debates on the subject of congestion [6-9, 13], most of them arise due to the fact that the different models proposed to identify congestion work under different assumptions.

In the existing literature on DEA, many models have been introduced to identify the input congestion of DMUs. Among these models, the most famous one, which is called the one-model approach, was introduced by Cooper *et al.* [11] and further expanded into the BCC input relaxation model. On the other hand, uncertainty is inevitable and commonly exists in many decision-making problems. For this reason, the fuzzy set theory has been proposed as a way to quantify imprecise and vague data in DEA models.

In this paper, we proposed a BCC input relaxation model with precise and imprecise data for determining the input congestion of DMUs in the mathematical programming technique of DEA with the one-model approach that has been introduced by Cooper *et al.* [11]. Furthermore, the possibility theory is used to

determine the possibilities of the fuzzy constraints. We have solved the fuzzy models by using the concept of chance-constrained programming, which has been introduced by Charnes and Cooper in 1959 [5].

The proposed one-model approach was further applied to determine the input congestion of 16 hospitals, wherein each hospital used four fuzzy inputs to produce two fuzzy outputs. These fuzzy inputs and outputs have symmetrical triangular membership functions.

Identifying the input congestion with the input relaxation model and fuzzy inputs and outputs and solving it with the possibility approach is a new work that has been presented in this article.

For future extensions, how to develop the proposed model with fuzzy random data will be an important research issue. Finally, it is hoped that this research makes a contribution to the field of DEA.

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