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Optimum cost analysis for an $Geo/Geo/c/N$ feedback queue under synchronous working vacations and impatient customers

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Abstract. This paper concerns the cost optimisation analysis of a discrete-time finite-capacity multiserver queueing system with Bernoulli feedback, synchronous multiple and single working vacations, balking, and reneging during both busy and working vacation periods. A reneged customer can be retained in the system by employing certain persuasive mechanism for completion of service. Using recursive method, the explicit expressions for the stationary state probabilities are obtained. Various system performance measures are presented. Further, a cost model is formulated. Then, the optimization of the model is carried out using quadratic fit search method (QFSM). Finally, the impact of various system parameters on the performance measures of the queueing system is shown numerically.

Keywords: multiserver queueing systems, synchronous vacation, impatient customers, Bernoulli feedback, cost model, optimization

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1. Introduction

Discrete-time queueing systems attainted a significant importance because of their wide applicability in the performance analysis of telecommunication systems. They are very appropriate for modeling and analyzing digital communication systems. Typical examples are synchronous communication systems (slotted ALOHA), packet switching systems with time slots and broad integrated services digital networks (B-ISDN) based on asynchronous transfer mode (ATM) technology, as the information contained in the B-ISDN is routed through discrete units. More details on discrete-time queues are given in the survey paper of [10] and the monographs of [9, 21, 8].

Most of the literature on customers' impatience in vacation queues focus on the continuous-time queueing models. Compared to the continuous-time case, the discrete-time vacation/working vacation queues received less attention in literature. In fact, discrete-time queueing systems with impatient customers are better suited for the design and analysis of slotted time communication systems. Analysis and modeling of the discrete-time multiserver queueing systems with impatient customers is more absorbed and different than the equivalent continuous time counterpart. In diverse companies, revenue losses due to balking and reneging are enormous and

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must therefore be examined in an appropriate context. The study on the discrete-time vacation queues with customers' impatience seems to be limited. The readers can refer to [16, 6, 7].

Many practical working vacation queueing systems with impatient customers have been well applied to many real-life problems including situations involving impatient telephone switchboard customers, telecommunication systems, manufacturing and production systems. [30] presented the analysis of customers' impatience in an infinite-buffer single server Markovian queue with working vacation. [26] considered a $M/M/1/N$ working vacation queue with balking and reneging. [18] dealt with impatient customers in an $M/M/1$ queue with single and multiple working vacations. [27] studied working vacation queueing model with Bernoulli schedule vacation interruption and impatient customers. Later, [28] provided an analysis on a queueing model with variant working vacations and customers' impatience. The transience analysis of an infinte-buffer single server queue with working vacation, heterogeneous service and impatient customers has been done in [19]. Then, [3] provided a study of a queueing system with Bernoulli feedback, reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption.

Over the last decade, multiserver vacation queues have been studied by several researchers. The servers in these systems may either take the same vacation together (synchronous vacation) or individual vacations (asynchronous vacations) independently. Most multiserver vacation models are based on synchronous vacations. Under such policy, the servers go on vacation all together, once the system is empty, and return to the busy period as one after the vacation period is ended. It is worth noting that multiserver vacation queueing systems are more complex compared to single-server vacation queues. Hence, a limited literature is available for these models. [15] presented the early work on the multiserver vacation model. [31, 32] considered Markovian multiserver queueing systems with single and multiple synchronous vacations. Further, [23] dealt with a more general $GI/M/c$ queueing system with phase-type vacations where all servers take multiple vacations together until waiting customers exist at a vacation completion instant. Then, [29] provided the analysis of a $M/M/c/N$ queueing model with impatient customers and synchronous vacations of some partial servers. A multi-server queueing model with Markovian arrival and synchronous phase type vacations was investigated by [4], using probabilistic rule and controlled thresholds. Later, [5] treated a finite-source multi-server queueing models with single and multiple vacations, the authors developed some algorithms for computing performance measures of the models. Recently, [17] studied a $M/M/c$ queueing system with single and multiple synchronous working vacations. Then, [1] presented the cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and customers' impatience.

In this investigation, we consider a finite-capacity discrete-time multiserver queueing system with synchronous single and multiple working vacations, balking, reneging, and retention of reneged customers. Customers may rejoin the tail of the queue as feedback customers if the service is incomplete or unsatisfactory. It is worth noting that [20] is a pioneering work on feedback queues. Since then, queueing models with feedback have attracted a significant attention of many researchers, see for instance [22, 11, 24] and references therein. In addition, in most literature, it was supposed that customer's impatience occurs because of the absence of the server(s). While in the current work, we assume that customers may lose their impatience and leave the system during both busy and working vacation periods. This situation takes place when the status of the server(s) is not observable or when the customers are unhappy with the service time. This is a very realistic supposition and we often encounter such queueing situations in real world phenomena, such as call center, where a call can be abandoned if waiting customers run out of patience. Further, as customer's impatience has became a challenging task, we incorporate in this paper the concept of customer retention which assumes a substantial importance for the business management. Firms can use certain mechanisms in order to convince impatient customers to stay in system in order to maintain their businesses, this can be either by increasing the service rates or providing more service channels in the system. This interesting idea was the subject of interest of some researchers including [12, 14, 13, 3, 2]. For the queueing system under consideration, we use a recursive method in order to obtain the steady state solution of the model. Then, we derive useful performance measures of the queueing system and formulate a cost model. After that, we perform a sensitivity numerical analysis. To the best of our knowledge, no similar work has been investigated in the literature, despite the fact that some of the aspects presented above have been discussed separately earlier.

The rest of the paper is arranged as follows: In Section 2, we describe the model. In Section 3, we carry out the stationary-state probabilities of the queueing system under consideration. Then, in Section 4, various performance measures of the queueing system are derived. In Section 5, we develop a cost model and perform a suitable optimization using a quadratic fit search method (QFSM). Then, in Section 6, we present numerical study to show the applicability of the theoretical analysis. Finally, Section 7 concludes the paper.

2. Model's description

Consider a finite-buffer discrete-time multiserver queueing system with Bernoulli feedback, single and multiple working vacation policies, balking, reneging during both normal busy and working vacation periods, and retention of reneged customers under late arrival system with delayed access (LASDA).

The time axis time is divided into fixed equal length intervals or slots with the length of a slot being unity, and it is indicated as 0, 1, 2,..,t,.... A potential arrival occurs in $(t-, t)$ and a potential departure takes place in $(t, t+)$. Let $\overline{x} = 1 - x$.

We suppose that the inter-arrival times A of customers are independent and geometrically distributed with probability mass function (p.m.f.):

$$a_n = P(A = n) = \lambda\overline{\lambda}^{n-1}, \;\; n \geq 1, \;\; 0 < \lambda < 1.$$

The system is composed of c servers. The customers are served on a First-Come First-Served discipline. The capacity of the system is taken as finite (say, N). A queue gets developed when the number of customers surpasses the number of servers, that is, when $n > c$.

If on arrival, a customer finds some servers busy and some others idle, he joins the system and will be served immediately. Otherwise, if all the servers are busy either during normal busy period or working vacation period, the arriving customer may join the queue with probability ϑ_n or balk with a complementary probability $\overline{\vartheta}_n = 1 - \vartheta_n$, with $c \leq n \leq N$. In addition, we suppose that

$$0 \leq \vartheta_{n+1} \leq \vartheta_n \leq 1, \;\; c \leq n \leq N-1, \;\; \vartheta_0 = 1, ..., \vartheta_{c-1} = 1, \;\; \text{and} \;\; \vartheta_N = 0.$$

The service times in normal busy period S_μ are independent and geometrically distributed with probability mass function (p.m.f.):

$$P(S = n) = \mu\,\overline{\mu}^{n-1}, \;\; n \geq 1, \;\; 0 < \mu < 1.$$

The service times S_η in a working vacation period are independent and geometrically distributed with probability mass function (p.m.f.):

$$P(S_\eta = n) = \eta\,\overline{\eta}^{n-1}, \;\; n \geq 1, \;\; 0 < \eta < 1.$$

A synchronous vacation is considered, that is, the servers go on working vacation all together at the same time once the system becomes empty, and they also return to the system as one at the same time.

Under multiple working vacation policy (MWV), the servers begin working vacation all together when the system becomes empty. If they return from the working vacation period and find the system empty they begin another vacation period. Otherwise, a normal busy period begins.

Under single working vacation policy (SWV), the servers take a working vacation once the system becomes empty. When this period is ended, they come back to the system as one and wait for customers to arrive rather than taking another work working vacation. Otherwise, they begin a new normal busy period.

The vacation times V are independent and geometrically distributed with probability mass function (p.m.f.):

$$P(V=n)=\theta\,\overline{\theta}^{n-1},\ \ n\geq 1,\ \ 0<\theta<1.$$

After joining the queue each customer will wait a certain length of time T for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting service with some probability σ. Using certain convincing mechanism, the reneged customer can be retained in the queueing system with probability $\overline{\sigma}=1-\sigma$. The impatience timer T is independent and geometrically distributed with probability mass function (p.m.f.):

$$P(T=n)=\xi\,\overline{\xi}^{n-1},\ \ n\geq 1,\ \ 0<\xi<1.$$

After completion of each service, a customer can either join the end of the queue for another regular service with probability $\overline{\beta}$ or leave the system with probability β, where $\overline{\beta}=1-\beta$.

It is worth noting that at each slot, only one arrival and/or one departure from service may occur. Which neglect any hypothesis stipulating that at one slot, we can have i services completed given j servers busy, $j=0,...,c$ and $i=0,...,j$. The same for the reneging phenomenon, we cannot have at the same time k reneged customers given that n customers are waiting in the queue, $n=1,...,N-c$ and $k=1,...,n$.

Further, note that at one slot we may have, one arrival, a departure from a service, and a departure from the queue as reneged customer.

3. Steady-state analysis

In this section, we establish the analysis of the considered queueing system under both single and multiple working vacations. Let δ be the indicator function, so

$$\delta=\begin{cases}1, \text{ for the single working vacation model,}\\ 0, \text{ for the multiple working vacation model.}\end{cases}$$

At steady-state, $\pi_{i,0},\ 0\leq i\leq N$ denotes the probability that there are i customers in the system when the servers are in working vacation period and $\pi_{i,1},\ 1-\delta\leq i\leq N$ is the probability that there are i customers in the system when the servers are in normal busy period.

Based on the one-step transition analysis, the steady-state equations can be given as

$$\begin{aligned}\pi_{0,0}=\ &\overline{\lambda}(1-\theta\delta)\pi_{0,0}+\ \overline{\lambda\vartheta}_1 d_1(\eta)\pi_{1,0}+\ \overline{\lambda\vartheta}_2 d_2(\eta)r_2\pi_{2,0}+\ \overline{\lambda\vartheta}_1 d_1(\mu)\pi_{1,1}\\ &+\overline{\lambda\vartheta}_2 d_2(\mu)r_2\pi_{2,1},\end{aligned}\tag{1}$$

$$\begin{aligned}\pi_{i,0}=\ &\overline{\theta}\left[\left(\ \overline{\lambda\vartheta}_{i+1}(d_{i+1}(\eta)\ \overline{r}_{i+1}+\ \overline{d}_{i+1}(\eta)r_{i+1})+\lambda\vartheta_{i+1}d_{i+1}(\eta)r_{i+1}\right)\pi_{i+1,0}\right.\\ &+\left(\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\eta)\ \overline{r}_{i-1}\right)\pi_{i-1,0}+\ \overline{\lambda\vartheta}_{i+2}d_{i+2}(\eta)r_{i+2}\pi_{i+2,0}\\ &\left.\left(\overline{\lambda\vartheta}_i\ \overline{d}_i(\eta)\ \overline{r}_i+\lambda\vartheta_i(d_i(\eta)\ \overline{r_i}+\ \overline{d}_i(\eta)r_i)\right)\pi_{i,0}\right],\ \ 1\leq i\leq N-2,\end{aligned}\tag{2}$$

$$\begin{aligned}\pi_{N-1,0} = \ & \overline{\theta}\Big[\big(\ \overline{\lambda\vartheta}_{N-1}\ \overline{d}_{N-1}(\eta)\ \overline{r}_{N-1} + \lambda\vartheta_{N-1}(d_{N-1}(\eta)\ \overline{r}_{N-1} + \overline{d}_{N-1}(\eta) \\ & r_{N-1})\big)\pi_{N-1,0} + \ \lambda\vartheta_{N-2}\ \overline{d}_{N-2}(\eta)\ \overline{r}_{N-2}\pi_{N-2,0}\left(d_N(\eta)\overline{r}_N\right. \\ & \left.+\overline{d}_N(\eta)r_N\right)\pi_{N,0}\Big], \end{aligned} \tag{3}$$

$$\pi_{N,0} = \ \overline{\theta}\ \overline{d}_N(\eta)\overline{r}_N\pi_{N,0} + \ \overline{\theta}\lambda\vartheta_{N-1}\ \overline{d}_{N-1}(\eta)\overline{r}_{N-2}\pi_{N-1,0}, \tag{4}$$

$$\pi_{0,1} = \overline{\lambda}\pi_{0,1} + \theta\overline{\lambda}\delta\pi_{0,0}, \tag{5}$$

$$\begin{aligned}\pi_{1,1} = & \big(\ \overline{\lambda\vartheta}_1\ \overline{d}_1(\mu) + \lambda\vartheta_1 d_1(\mu)\big)\pi_{1,1} + \big(\ \overline{\lambda\vartheta}_2(d_2(\mu)\ \overline{r}_2 + \overline{d}_2(\mu)r_2) + \lambda\vartheta_2 d_2(\mu) \\ & r_2\big)\pi_{2,1} + \ \overline{\lambda\vartheta}_3 d_3(\mu)r_3\pi_{3,1} + \lambda\delta\pi_{0,1} + \theta\big(\ \overline{\lambda\vartheta}_1\ \overline{d}_1(\eta) + \lambda\vartheta_1 d_1(\eta)\big)\pi_{1,0} \\ & +\theta\lambda\pi_{0,0} + \theta\big(\ \overline{\lambda\vartheta}_2(d_2(\eta)\ \overline{r}_2 + \ \overline{d}_2(\eta)r_2) + \lambda\vartheta_2 d_2(\eta)r_2\big)\pi_{2,0} + \theta\overline{\lambda\vartheta}_3 \\ & d_3(\eta)r_2\pi_{3,0}, \end{aligned} \tag{6}$$

$$\begin{aligned}\pi_{i,1} = & \big(\ \overline{\lambda\vartheta}_i\ \overline{d}_i(\mu)\ \overline{r}_i + \lambda\vartheta_i(d_i(\mu)\ \overline{r}_i + \ \overline{d}_i(\mu)r_i)\big)\pi_{i,1} + \lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\overline{r}_{i-1} \\ & \times\ \pi_{i-1,1} + \big(\ \overline{\lambda\vartheta}_{i+1}(d_{i+1}(\mu)\ \overline{r}_{i+1} + \ \overline{d}_{i+1}(\mu)r_{i+1}) + \lambda\vartheta_{i+1}d_{i+1}(\mu)r_{i+1}\big) \\ & \times\pi_{i+1,1} + \ \overline{\lambda\vartheta}_{i+2}d_{i+2}(\mu)r_{i+2}\pi_{i+2,1} + \theta\big(\ \overline{\lambda\vartheta}_i\ \overline{d}_i(\eta)\ \overline{r}_i + \lambda\vartheta_i(d_i(\eta)\ \overline{r}_i \\ & +\ \overline{d}_i(\eta)r_i)\big)\pi_{i,0} + \theta\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\eta)\ \overline{r}_{i-1}\pi_{i-1,0} + \theta\big(\ \overline{\lambda\vartheta}_{i+1}(d_{i+1}(\eta)\overline{r}_{i+1} \\ & +\ \overline{d}_{i+1}(\eta)r_{i+1}) + \lambda\vartheta_{i+1}d_{i+1}(\eta)r_{i+1}\big)\pi_{i+1,0} + \ \overline{\lambda\vartheta}_{i+2}d_{i+2}(\eta)r_{i+2}\pi_{i+2,0}\theta, \end{aligned} \tag{7}$$

$$\begin{aligned}\pi_{N-1,1} = & \big(\ \overline{\lambda\vartheta}_{N-1}\ \overline{d}_{N-1}(\mu)\ \overline{r}_{N-1} + \lambda\vartheta_{N-1}\left(d_{N-1}(\mu)\ \overline{r}_{N-1} + \overline{d}_{N-1}(\mu)\right. \\ & \left.\times r_{N-1})\right)\pi_{N-1,1} + \lambda\vartheta_{N-2}\ \overline{d}_{N-2}(\mu)\ \overline{r}_{N-2}\pi_{N-2,1} + (d_N(\mu)\ \overline{r}_N \\ & -\ \overline{d}_N(\mu)r_N\big)\pi_{N,1} + \theta(\ \overline{\lambda\vartheta}_{N-1}\overline{d}_N(\eta)\ \overline{r}_{N-1}) + \lambda\vartheta_{N-1}(d_{N-1}(\eta)\ \overline{r}_{N-1} \\ & +\ \overline{d}_{N-1}(\eta)r_{N-1}))\pi_{N-1,0} + \theta\lambda\vartheta_{N-2}\ \overline{d}_{N-2}(\eta)\ \overline{r}_{N-2}\pi_{N-2,0} \\ & +\theta\left(d_N(\eta)\ \overline{r}_N - \ \overline{d}_N(\eta)r_N\right)\pi_{N,0}, \end{aligned} \tag{8}$$

$$\begin{aligned}\pi_{N,1} = \ & \overline{d}_N(\mu)\ \overline{r}_N\pi_{N,1} + \lambda\vartheta_{N-1}\ \overline{d}_{N-1}(\mu)\ \overline{r}_{N-1}\pi_{N-1,1} + \theta\ \overline{d}_N(\eta)\ \overline{r}_N\pi_{N,0} \\ & +\theta\lambda\vartheta_{N-1}\ \overline{d}_{N-1}(\eta)\ \overline{r}_{N-1}\pi_{N-1,0}, \end{aligned} \tag{9}$$

where

$$d_n(x) = \begin{cases} 1-\overline{x\beta}^n, & \text{if } 1 \le n \le c-1, \\ 1-\ \overline{x\beta}^c, & \text{if } c \le n \le N, \end{cases}$$

and

$$r_n = \begin{cases} 0, & \text{if } 1 \le n \le c, \\ 1-\ \overline{\xi\sigma}^{n-c}, & \text{if } c+1 \le n \le N. \end{cases}$$

This can be explicable by the fact that a service occurs for one of n customers in the service system with probability $1-\overline{x\beta}^n$, when n customers are being served by c servers such that $1 \le n \le c-1$. In the case where the number of customers in the whole system exceeds the number of present servers, the probability that a service will take place is given by $1-\overline{x\beta}^c$.

In the same manner, when the number of customers in the system is less than the number of servers, the probability that an impatience phenomenon occurs is equal to 0, contrariwise to the case where the number of customers in the system is bigger that the number of servers present in the system, the probability that a customer leaves the system is equal to $1-\overline{\xi\sigma}^{n-c}$.

We obtain the steady-state probabilities $\pi_{i,0},\ 0\le i\le N$ and $\pi_{i,1},\ 1-\delta\le i\le N$, using a recursive method. Solving Equations (2)-(4) recursively, we find

$$\pi_{i,0}=\varrho_i\pi_{N,0},\ \ 0\le i\le N, \tag{10}$$

with

$$\begin{aligned}\varrho_0=&\left(\left(1-\ \overline{\theta}\left(\ \overline{\lambda\vartheta}_1\ \overline{d}_1(\eta)\right)\right)\varrho_1-\ \overline{\theta}\ \overline{\lambda\vartheta}_3 d_3(\eta)r_3\varrho_3-\overline{\theta}\left(\ \overline{\lambda\vartheta}_2\left(d_2(\eta)\ \overline{r}_2\right.\right.\right.\\&\left.\left.\left.+\ \overline{d}_2(\eta)\overline{r}_2\right)\right)\varrho_2\right)/(\ \overline{\theta}\lambda),\end{aligned} \tag{11}$$

$$\begin{aligned}\varrho_i=&\left(\left(1-\ \overline{\theta}\left[\ \overline{\lambda\vartheta}_{i+1}\ \overline{d}_{i+1}(\eta)\ \overline{r}_{i+1}+\lambda\vartheta_{i+1}\left(d_{i+1}(\eta)\overline{r}_{i+1}+\overline{d}_{i+1}(\eta)\right.\right.\right.\right.\\&\left.\left.\left.\times r_{i+1})\right]\right)\varrho_{i+1}-\ \overline{\theta}\ \overline{\lambda\vartheta}_{i+2}\left(d_{i+2}(\eta)\ \overline{r}_{i+2}+\ \overline{d}_{i+2}(\eta)\ \ r_{i+2}\right)\varrho_{i+2}\\&\left.-\ \overline{\theta}\ \overline{\lambda\vartheta}_{i+3}d_{i+3}(\eta)r_{i+3}\varrho_{i+3}\right)/\left(\ \overline{\theta}\lambda\vartheta_i\ \overline{d}_i(\eta)\ \overline{r}_i\right),\ \ i=N-3,...,1,\end{aligned} \tag{12}$$

$$\begin{aligned}\varrho_{N-2}=&\left(\left(\left(1-\ \overline{\theta}\left[\ \overline{\lambda\vartheta}_{N-1}\ \overline{d}_{N-1}(\eta)\ \overline{r}_{N-1}+\lambda\vartheta_{N-1}\left(d_{N-1}(\eta)\overline{r}_{N-1}+r_{N-1}\right.\right.\right.\right.\right.\\&\left.\left.\left.\left.\overline{d}_{N-1}(\eta)\right)\right]\right)\varrho_{N-1}-\ \overline{\theta}\ \overline{d}_N(\eta)\ \overline{r}_N\varrho_N\right)\left(\overline{\theta}\lambda\vartheta_{N-2}\ \overline{d}_{N-2}(\eta)\overline{r}_{N-2}\right)^{-1},\end{aligned} \tag{13}$$

$$\varrho_{N-1}=\left(1-\ \overline{\theta}\ \overline{d}_N(\eta)\ \overline{r}_N\right)/\ \left(\overline{\theta}\lambda\vartheta_{N-1}\ \overline{d}_{N-1}(\eta)\ \overline{r}_{N-1}\right), \tag{14}$$

and

$$\varrho_N=1. \tag{15}$$

Now, substituting Equation (10) in Equations (5) and (7)-(9), we get

$$\pi_{i,1}=\rho_i\pi_{N,1}+\varphi_i\pi_{N,0},\ \ 1-\delta\le i\le N, \tag{16}$$

where

$$\rho_0=0\ \ \varphi_0=\left(\frac{\theta\overline{\lambda}\delta\varrho_0}{\lambda}\right), \tag{17}$$

$$\begin{aligned}\rho_{i-1}=&\frac{\left(\ \overline{\lambda\vartheta}_{i+1}\left(d_{i+1}(\mu)\ \overline{r}_{i+1}+\ \overline{d}_{i+1}(\mu)\ r_{i+1}\right)+\lambda\vartheta_{i+1}d_{i+1}(\mu)r_{i+1}\right)\rho_{i+1}}{\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}}\\&+\frac{\left(1-\left[\ \overline{\lambda\vartheta}_i\ \overline{d}_i(\mu)\ \overline{r}_i+\lambda\vartheta_i\left(d_i(\mu)\overline{r}_i+\ \overline{d}_i(\mu)r_i\right)\right]\right)\rho_i}{\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}}\\&-\frac{\overline{\lambda\vartheta}_{i+2}d_{i+2}(\mu)r_{i+2}\rho_{i+2}}{\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}},\ \ i=N-3,...,2,\end{aligned} \tag{18}$$

$$\varphi_{i-1} = \frac{\left(1 - \left[\ \overline{\lambda\vartheta}_i\ \overline{d}_i(\mu)\ \overline{r}_i + \lambda\vartheta_i\left(d_i(\mu)\overline{r}_i + \ \overline{d}_i(\mu)r_i\right)\right]\right)\varphi_i}{\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}}$$

$$-\frac{\overline{\lambda\vartheta}_{i+1}\left(d_{i+1}(\mu)\ \overline{r}_{i+1} + \ \overline{d}_{i+1}(\mu)\ r_{i+1}\right)\varphi_{i+1}}{\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}}$$

$$-\theta\frac{\left[\ \overline{\lambda\vartheta}_i\ \overline{d}_i(\eta)\ \overline{r}_i + \lambda\vartheta_i\left(d_i(\eta)\overline{r}_i + \ \overline{d}_i(\eta)r_i\right)\right]\varrho_i}{\left(\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}\right)}$$

$$-\frac{\theta\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\eta)\ \overline{r}_{i-1}\varrho_{i-1} - \ \overline{\lambda\vartheta}_{i+2}d_{i+2}(\mu)r_{i+2}\varphi_{i+2}}{\left(\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}\right)} \tag{19}$$

$$+\frac{\theta\ \overline{\lambda\vartheta}_{i+2}d_{i+2}(\eta)r_{i+2}\varrho_{i+2} + \lambda\vartheta_{i+1}d_{i+1}(\mu)r_{i+1}\varphi_{i+1}}{\left(\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}\right)}$$

$$+\frac{\theta\ \overline{\lambda\vartheta}_{i+1}\left(d_{i+1}(\eta)\ \overline{r}_{i+1} + \ \overline{d}_{i+1}(\eta)\ r_{i+1}\right)\varrho_{i+1}}{\left(\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}\right)}$$

$$+\frac{\theta\lambda\vartheta_{i+1}d_{i+1}(\eta)r_{i+1}\varrho_{i+1}}{\left(\lambda\vartheta_{i-1}\ \overline{d}_{i-1}(\mu)\ \overline{r}_{i-1}\right)},\ \ i = N-3, ..., 2,$$

$$\begin{aligned}\rho_{N-2} = &\left[1 - \left[\ \overline{\lambda\vartheta}_{N-1}\ \overline{d}_{N-1}(\mu)\ \overline{r}_{N-1} + \lambda\vartheta_{N-1}\left(\overline{d}_{N-1}(\mu)r_{N-1}\right.\right.\right.\\ &\left.\left.+\ \overline{d}_{N-1}(\mu)r_{N-1}\right)\right]\rho_{N-1} - \left(d_N(\mu)\ \overline{r}_N + \ \overline{d}_N(\mu)r_N\right)\rho_N\Big]\\ &\times\left(\lambda\vartheta_{N-2}\ \overline{d}_{N-2}(\mu)\ \overline{r}_{N-2}\right)^{-1},\end{aligned} \tag{20}$$

$$\begin{aligned}\varphi_{N-2} = &\left[\left(1 - \left(\ \overline{\lambda\vartheta}_{N-1}\ \overline{d}_{N-1}(\mu)\ \overline{r}_{N-1} + \lambda\vartheta_{N-1}\left(d_{N-1}(\mu)\overline{r}_{N-1} + \overline{d}_{N-1}(\mu)\right.\right.\right.\right.\\ &\left.\left.\left. r_{N-1}\right)\right)\right)\varphi_{N-1} - \theta\left(\lambda\vartheta_{N-2}\ \overline{d}_{N-2}(\eta)\ \overline{r}_{N-2}\varrho_{N-2} + \left(\ \overline{\lambda\vartheta}_{N-1}\ \overline{d}_{N-1}(\eta)\right.\right.\\ &\left.\overline{r}_{N-1} + \lambda\vartheta_{N-1}\left(d_{N-1}(\eta)\overline{r}_{N-1} + \ \overline{d}_{N-1}(\eta)r_{N-1}\right)\right)\varrho_{N-1} + d_N(\eta)\ \overline{r}_N\\ &\left.\left.+\ \overline{d}_N(\eta)r_N\right)\right] \times \left(\lambda\vartheta_{N-2}\ \overline{d}_{N-2}(\mu)\ \overline{r}_{N-2}\right)^{-1},\end{aligned} \tag{21}$$

$$\rho_{N-1} = \frac{1 - \ \overline{d}_N(\mu)\ \overline{r}_N}{\lambda\vartheta_{N-1}\ \overline{d}_{N-1}(\mu)\ \overline{r}_{N-1}}, \tag{22}$$

$$\varphi_{N-1} = -\theta\left(\frac{\overline{d}_N(\eta)\ \overline{r}_N\varrho_N + \lambda\vartheta_{N-1}\ \overline{d}_{N-1}(\eta)\ \overline{r}_{N-1}\varrho_{N-1}}{\lambda\vartheta_{N-1}\ \overline{d}_{N-1}(\mu)\ \overline{r}_{N-1}}\right), \tag{23}$$

$$\rho_N = 1,\ \text{ and }\ \varphi_N = 0. \tag{24}$$

Next, substituting Equations (10) and (16) in Equation (6), we obtain

$$\pi_{N,1} = \psi\pi_{N,0}, \tag{25}$$

where

$$\psi = \theta\frac{(\ \overline{\lambda\vartheta}_1\ \overline{d}_1(\eta) + \lambda\vartheta_1 d_1(\eta))\varrho_1 +\ \overline{\lambda\vartheta}_3 d_3(\eta) r_3\varrho_3 + \lambda\varrho_0 + \overline{\lambda\vartheta}_2(d_2(\eta)\ \overline{r}_2\varrho_2}{\grave{\psi}}$$

$$+\frac{\theta\overline{d}_2(\eta)r_2\varrho_2 + (\ \overline{\lambda\vartheta}_2(d_2(\mu)\ \overline{r}_2 + \overline{d}_2(\mu)r_2))\varphi_2 + \overline{\lambda\vartheta}_3 d_3(\mu)r_3\varphi_3 + \delta\lambda\varphi_0}{\grave{\psi}} \tag{26}$$

$$-\frac{\left(\left(1-\ \overline{\lambda\vartheta}_1\ \overline{d}_1(\mu) + \lambda\vartheta_1 d_1(\mu)\right)\right)\varphi_1}{\grave{\psi}},$$

and

$$\grave{\psi} = -\left(\ \overline{\lambda\vartheta}_2(d_2(\mu)\ \overline{r}_2 + \overline{d}_2(\mu)r_2)\right)\rho_2 -\ \overline{\lambda\vartheta}_3 d_3(\mu)r_3\rho_3 + (1-(\ \overline{\lambda\vartheta}_1\ \overline{d}_1(\mu) + \lambda\vartheta_1 d_1(\mu)))\rho_1.$$

Finally, $\pi_{N,0}$ is obtained using the normalization condition

$$\sum_{i=0}^{N}\pi_{i,0} + \sum_{i=1-\delta}^{N}\pi_{i,1} = 1.$$

Therefore,

$$\pi_{N,0} = \left(\sum_{i=0}^{N}\varrho_i + \sum_{i=1-\sigma}^{N}(\varphi_i + \psi\rho_i)\right)^{-1}. \tag{27}$$

4. Performance measures

− The average system size ($E(L)$).

$$E(L) = E(L_0) + E(L_1), \tag{28}$$

where ($E(L_0)$) (resp. ($E(L_1)$)) denotes the average system size when the servers are on working vacation period (resp. on normal busy period).

$$E(L_0) = \sum_{i=1}^{N} i\pi_{i,0} \ \text{ and }\ E(L_1) = \sum_{i=1}^{N} i\pi_{i,1}. \tag{29}$$

− The average queue length ($E(L_q)$).

$$E(L_q) = \sum_{i=c+1}^{N}(i-c)\pi_{i,0} + \sum_{i=c+1}^{N}(i-c)\pi_{i,1}. \tag{30}$$

− The average number of customers served per unit time (S_r).

$$\begin{aligned} S_r &= \beta\sum_{i=1}^{N} i\left(\eta\pi_{i,0} + \mu\pi_{i,1}\right) \\ &= \beta\sum_{i=1}^{c-1} i\left(\eta\pi_{i,0} + \mu\pi_{i,1}\right) + c\beta\sum_{i=c}^{N}\left(\nu\pi_{i,0} + \mu\pi_{i,1}\right). \end{aligned} \tag{31}$$

– The probabilities that the servers are on normal busy period (P_b), on working vacation period (P_{wv}), and idle during busy period. (P_{id}), respectively.

$$P_b = \sum_{i=1}^{N} \pi_{i,1}, \quad P_{wv} = \sum_{i=0}^{N} \pi_{i,0}, \quad \text{and} \quad P_{id} = \delta\pi_{0,1}. \tag{32}$$

– The average balking rate (B_r).

$$B_r = \lambda \sum_{i=1}^{N} (1-\vartheta_i)(\pi_{i,0} + \pi_{i,1}). \tag{33}$$

– The average reneging rate (R_{ren}).

$$R_{ren} = \sigma\xi \sum_{i=c+1}^{N} (i-c)(\pi_{i,0} + \pi_{i,1}). \tag{34}$$

– The average retention rate (R_{ret}).

$$R_{ret} = (1-\sigma)\xi \sum_{i=c+1}^{N} (i-c)(\pi_{i,0} + \pi_{i,1}). \tag{35}$$

– The average rate of customer loss (R_{loss}).

$$R_{loss} = B_r + R_{ren}. \tag{36}$$

5. Cost model

To develop a cost model, we introduce the following cost elements

- C_b : Cost per unit time when the servers are in normal busy period.
- C_{wv} : Cost per unit time when the servers are on working vacation period.
- C_{id} : Cost per unit time when the servers are idle during busy period.
- C_q : Cost per unit time when a customer joins the queue and waits for service.
- C_{Rb} : Cost per unit time when a customer balks.
- C_{s1} : Cost per service per unit time when the servers are in normal busy period.
- C_{s2} : Cost per service per unit time when the servers are in working vacation period.
- C_{ren} : Cost per unit time when a customer reneges.
- C_{ret} : Cost per unit time when a customer is retained.
- C_{s-f} : Cost per unit time when a customer returns to the system as a feedback customer.
- C_a : Fixed server purchase cost per unit.

Using the above cost parameters, the total expected cost per unit time of the system, Γ, is given as

$$\Gamma = C_b P_b + C_{wv} P_{wv} + C_q E(L_q) + C_{Rb} B_r + C_r R_{ren} + C_{id} P_{id} + C_{ret} R_{ret} + c(\mu C_{s1} + \nu C_{s2})$$
$$+ c(\mu + \nu)(1-\beta) C_{s-f} + cC_a.$$

5.1. Quadratic fit search method

This part concerns the cost optimization problem under a given cost structure using quadratic fit search method (QFSM). The objective is to determine the optimal service rate during normal busy period, μ^*, in different cases, in order to minimize the expected cost function Γ. Suppose that all system parameters have fixed values, and the only controlled parameter is the service rate during normal busy period μ.

The cost minimization problem can be given as

$$\min_{\mu} \Gamma(\mu).$$

As it has been discussed in Laxmi et al. [25], given a 3-point pattern, we can fit a quadratic function by using corresponding functional values that has a unique minimum, x^q, for the given objective function $\Gamma(x)$. The unique optimum x^q of the quadratic function agreeing with $\Gamma(x)$ at 3-point operation (x^l, x^m, x^u) is given by

$$x^q \cong \frac{1}{2}\left[\frac{\Gamma(x^l)((x^m)^2-(x^u)^2)+\Gamma(x^m)((x^u)^2-(x^l)^2)+\Gamma(x^u)((x^l)^2-(x^m)^2)}{\Gamma(x^l)(x^m-x^u)+\Gamma(x^m)(x^u-x^l)+\Gamma(x^u)(x^l-x^m)}\right].$$

6. Numerical results

In this section, we carry out the optimization of the considered queueing system, using QFSM to minimize the expected cost function Γ with respect to the service rate during normal busy period, μ, based on the changes of others system parameters. In addition, we study the influence of the system parameters on various performance measures of the queueing model under multiple and single working vacation policies.

6.1. Optimisation study

In order to perform an economic study of the queueing model, we consider $\vartheta_n = 1 - \dfrac{n}{N}$, and fixe $C_b = 1,\ C_{wv} = 0.5,\ C_q = 1.5,\ C_{Rb} = 1,\ C_{ren} = 1,\ C_{id} = 0.5,\ C_{ret} = 1,\ C_{s1} = 2.5,\ C_{s2} = 2,\ C_{s-f} = 1$, and $C_a = 0.5$.

We display Tables 1-4 and Figures 1-4 to illustrate the optimal service values of μ, the optimum expected cost $\Gamma(\mu^*)$, as well as to show the convexity of the curves $\Gamma(\mu)$ for different values of ν, θ, ξ, and c. To this end, we consider the following cases

- Table 1 and Figure 1: $\lambda = 0.8$, $\beta = 0.7$, $c = 3$, $\theta = 0.4$, $\xi = 0.5$, $\alpha = 0.5$, and $N = 20$.
- Table 2 and Figure 2: $\lambda = 0.8$, $\beta = 0.7$, $c = 2$, $\nu = 0.2$, $\xi = 0.8$, $\alpha = 0.5$, and $N = 20$.
- Table 3 and Figure 3: $\lambda = 0.8$, $\beta = 0.7$, $c = 3$, $\theta = 0.4$, $\nu = 0.3$, $\alpha = 0.5$, and $N = 20$.
- Table 4 and Figure 4: $\lambda = 0.8$, $\beta = 0.7$, $\nu = 0.3$, $\theta = 0.4$, $\xi = 0.5$, $\alpha = 0.5$, and $N = 20$.

	$\nu = 0.05$		$\nu = 0.1$		$\nu = 0.15$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ^*	0.3530611	0.3529851	0.3532569	0.3531987	0.3534304	0.3533881
Γ^*	7.602	7.602198	7.946371	7.94651	8.29081	8.290895

Table 1: *The optimal values μ^* and Γ^*, for different values of ν, under SWV and MWV policies.*

– Using QFSM, the optimal values of μ and the minimum expected cost $F(\mu^*)$ are shown in Tables 1-4, for different values of ν, θ, ξ, and c respectively. From Figures 1-4, it is well observed the convexity of the curves for different values of ν, θ, ξ, and c. This proves that there

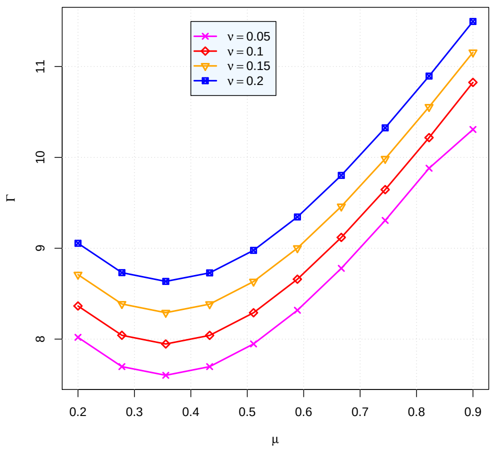

Figure 1: *The optimum service rate μ^*, for different values of ν, under SWV policy.*

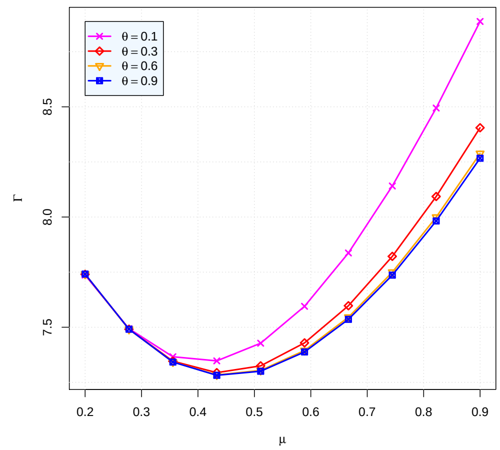

Figure 2: *The optimum service rate μ^*, for different values of θ, under SWV policy.*

	$\theta = 0.1$		$\theta = 0.3$		$\theta = 0.9$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ^*	0.4082898	0.4075521	0.4418178	0.4411087	0.4526102	0.4525709
Γ^*	7.342124	7.343256	7.293472	7.294413	7.279612	7.279648

Table 2: *The optimal values μ^* and Γ^*, for different values of θ, under SWV and MWV policies.*

exists a certain value of the service rate μ that minimizes the total expected cost function for the chosen set of system parameters.

– From Tables 1-3, we observe that for different values of ν, θ, and ξ, the minimum expected cost $\Gamma(\mu^*)$ in SWV model is lower than that in MWV model, as intuitively expected. While from Table 4, $\Gamma(\mu^*)$ in SWV model is larger than that in MWV model. This can be explained by the fact that for $c = 2, 4$, the optimum service rate μ^* under SWV policy is smaller than μ^* under MWV policy. In addition, this can be because of the choice of the system parameters.

6.2. Performance study

In this subsection, different performance measures of interest computed under different scenarios are presented. These measures are obtained by developing a program in R software. To illustrate the system numerically, we take $\vartheta_n = 1 - \dfrac{n}{N}$, and consider the following cases

- Tables 5-6: $\lambda = 0.7$, $\beta = 0.6$, $\theta = 0.2$, $\nu = 0.4$, $\mu = 0.8$, $\xi = 0.8$, $\alpha = 0.8$, $c = 2$, $N = 20$, unless they are considered as variables.

- Table 7: $\lambda = 0.7$, $\beta = 0.6$, $\theta = 0.2$, $\nu = 0.4$, $\mu = 0.5$, $\xi = 0.8$, $\alpha = 0.8$, $c = 2 : 2 : 4$, and $N = 20$.

1. According to Tables 5 and 6, for both MWV and SWV models, we have

   – Along the increasing of λ (resp. μ, ν, and β), the characteristics (P_b), $(E(L))$, (B_r), (R_{ren}), and (R_{ret}) increase (resp. decrease). While (P_{wv}) decrease (resp. increase) with λ (resp. with μ). This is quite reasonable, when the arrival rate increases, the average system size increases. This increases the probability that the servers are in normal busy

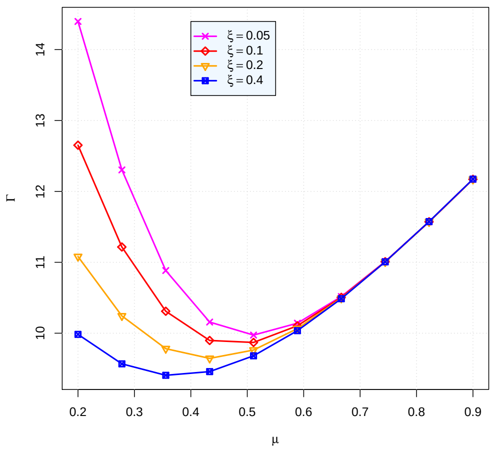

Figure 3: *The optimum service rate μ^*, for different values of ξ under MWV policy.*

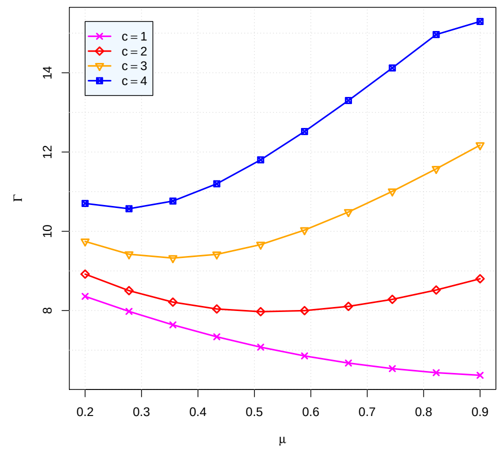

Figure 4: *The optimum service rate μ^*, for different values of c, under MWV policy.*

	$\xi = 0.1$		$\xi = 0.2$		$\xi = 0.4$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ^*	0.4777952	0.4777702	0.4325604	0.4325421	0.3733920	0.3733847
Γ^*	9.842556	9.842644	9.644805	9.644819	9.400941	9.400905

Table 3: *The optimal values μ^* and Γ^*, for different values of ξ, under SWV and MWV policies.*

period, which increases the average balking rate and the average reneging rate. Therefore, in order to keep the system size under control and to avoid more reneging of customers, the firm uses some strategies in order to maintain the impatient customer in the system. Thus, the average retention rate increases. Further, it is quite obvious that with increase in the service rates, the expected system size decreases which decreases the probability that the servers are in normal busy period and increases the probability the servers are on working vacation. Consequently, average balking and reneging rates decrease. In addition, along the increasing of the service rates, customers have to spend less time in the system, and there will be less possibility that customers get impatient and leave the system without getting service, which is a desirable condition for any firm. Then, the retention rate is decreased. Further, as intuitively expected, the average number of customers served (S_r) increases with λ, μ, ν, and β.

− With the increasing of θ, the probability that the servers are in normal busy period (P_b) increases, while the probability the servers are on working vacation (P_{wv}) decreases, this decreases the mean system size $(E(L))$, which decreases the average rates of balking (B_r), reneging (R_{ren}), and retention (R_{ret}). Consequently, the average rate of customers served (S_r) increases with θ, as it should be.

− When the impatience rate increases ξ, both (P_b), $(E(L))$, and (B_r) decrease, while $(\pi_{0,0})$, (P_{wv}), (R_{ren}), and (R_{ret}) increase, as intuitively expected. Further, (S_r) decreases with the parameter ξ. This is because the impatience rate increases the average rate of reneging, which decreases the mean system size $(E(L))$. This results in the decreasing of the mean number of customers served.

− With the increasing values of N, the average system size increases which increases the probability that the servers are in normal busy period and decreases the probability that the servers are on working vacation period. Consequently, (S_r) increases with N. This

	c=2		c=3		c=4	
	SWV	MWV	SWV	MWV	SWV	MWV
μ^*	0.527004	0.527511	0.353839	0.353835	0.267373	0.267376
Γ^*	7.969997	7.969291	9.324464	9.324418	10.56634	10.56633

Table 4: *The optimal values μ^* and Γ^*, for different values of c, under SWV and MWV policies.*

μ		$\pi_{0,0}$	P_b	P_{wv}	$E(L)$	B_r	R_{ren}	R_{ret}	S_r
0.5	λ =0.1	0.675301	0.172145	0.827855	0.374858	0.000488	0.003533	0.000883	0.101480
	λ =0.5	0.103486	0.732620	0.267380	1.596121	0.030830	0.202896	0.050724	0.414755
	λ =0.9	0.003651	0.983596	0.016404	2.431647	0.104459	0.642058	0.160515	0.563425
0.8	λ =0.1	0.726536	0.109334	0.890666	0.307101	0.000329	0.001970	0.000492	0.103017
	λ =0.5	0.170674	0.559027	0.440973	1.315834	0.022156	0.107264	0.026816	0.495736
	λ =0.9	0.010580	0.952457	0.047543	2.036343	0.082444	0.319146	0.079786	0.836878
0.5	θ =0.1	0.028374	0.822989	0.177011	2.056236	0.063919	0.409947	0.102487	0.498933
	θ =0.5	0.030080	0.949851	0.050149	2.033224	0.062878	0.395296	0.098824	0.509357
	θ =0.9	0.030502	0.967143	0.032857	2.030222	0.062739	0.393914	0.098479	0.510245
0.8	θ =0.1	0.055921	0.651138	0.348862	1.777336	0.051089	0.250669	0.062667	0.635222
	θ =0.5	0.071795	0.880304	0.119696	1.625750	0.044114	0.166481	0.041620	0.700563
	θ =0.9	0.076388	0.917715	0.082285	1.600515	0.042924	0.157635	0.039409	0.705597
0.5	ξ =0.1	0.011725	0.958726	0.041274	3.299279	0.112203	0.166914	0.041729	0.562062
	ξ =0.5	0.024704	0.913387	0.086613	2.245779	0.071701	0.347937	0.086984	0.520107
	ξ =0.9	0.030446	0.893580	0.106420	1.997149	0.061387	0.411887	0.102972	0.501576
0.8	ξ =0.1	0.047891	0.831409	0.168591	2.133873	0.065723	0.064708	0.016177	0.744343
	ξ =0.5	0.059718	0.790623	0.209377	1.786042	0.051285	0.167099	0.041775	0.691493
	ξ =0.9	0.064751	0.773669	0.226331	1.672464	0.046321	0.210168	0.052542	0.669196
0.5	N=10	0.034244	0.880647	0.119353	1.918922	0.115157	0.317459	0.079365	0.489337
	N=15	0.031005	0.891683	0.108317	1.998649	0.081709	0.371070	0.092767	0.499775
	N=20	0.029280	0.897583	0.102417	2.042622	0.063306	0.401005	0.100251	0.505335
0.8	N=10	0.070539	0.754147	0.245853	1.608134	0.085869	0.161442	0.040360	0.643540
	N=15	0.066151	0.768898	0.231102	1.663640	0.060988	0.187629	0.046907	0.662967
	N=20	0.063745	0.777031	0.222969	1.694181	0.047280	0.202141	0.050535	0.673639
0.5	ν=0.1	0.026306	0.901597	0.098403	2.086260	0.065251	0.429493	0.107373	0.488228
	ν=0.2	0.027146	0.900390	0.099610	2.070667	0.064567	0.418720	0.104680	0.494284
	ν=0.4	0.029280	0.897583	0.102417	2.042622	0.063306	0.401005	0.100251	0.505335
0.8	ν=0.1	0.057003	0.786771	0.213229	1.783940	0.051301	0.259074	0.064768	0.641041
	ν=0.2	0.058933	0.783751	0.216249	1.751799	0.049883	0.237399	0.059350	0.652646
	ν=0.4	0.063745	0.777031	0.222969	1.694181	0.047280	0.202141	0.050535	0.673639
0.5	β=0.1	0.000689	0.997406	0.002594	2.925449	0.101261	1.060973	0.265243	0.098284
	β=0.5	0.020320	0.927691	0.072309	2.184470	0.069676	0.495448	0.123862	0.437229
	β=0.9	0.063359	0.790780	0.209220	1.692740	0.047236	0.199913	0.049978	0.668883
0.8	β=0.1	0.001839	0.993082	0.006918	2.794082	0.095890	0.954641	0.238660	0.155009
	β=0.5	0.046732	0.833701	0.166299	1.839203	0.054008	0.277827	0.069457	0.605747
	β=0.9	0.118720	0.607971	0.392028	1.376739	0.032325	0.083418	0.020854	0.795246

Table 5: *Effect of μ, λ, θ, ξ, N, ν, and β on performance measures for MWV models.*

results in the decreasing of (B_r). Further, (R_{ren}) and (R_{ret}) monotonically increase with N. Obviously, the larger the system capacity, the greater the number of customers present in the system and the higher the average rates of reneging and retention.

2. The probability ($\pi_{0,0}$) increases with the vacation rate θ under multiple vacation policy. While under single vacation policy, it decreases with θ when $\mu = 0.5$, when $\mu = 0.8$, its behaviour is not monotonic. This can be because of the choice of the choice of the system parameters.

3. It is worth noting that the probability that the servers are idle during busy period (P_{idle}) is null for MWV. This is quite clear; at the end of working vacation period, if the system is still empty, the servers return to the working vacation period. While for SWV, once the working vacation time is ended, the servers come back to the normal busy period and stay there waiting for a new arriving customers. In this case, it is well seen that (P_{idle}) increases with μ, θ, ξ, ν, and β, and decreases with the increasing of λ and N, as it should be.

4. From Table 7, it well observed that along the increasing the number of servers c, the mean queue length ($E(L_q)$), the average balking rate (B_r), the average rates of reneging (R_{ren}) and retention (R_{ret}) decrease. Whereas, when c augments, the average number of customers served (S_r) monotonically increases. Obviously, when the number servers increases in the system, the customers are served significantly, this results in decreasing in the mean number of customers in the queue. Therefore, the average rates of balking, reneging and retention decrease.

μ		$\pi_{0,0}$	P_{id}	P_b	P_{wv}	$E(L)$	B_r	R_{ren}	R_{ret}	S_r
0.5	λ=0.1	0.246909	0.444437	0.252875	0.302688	0.353707	0.000439	0.003051	0.000763	0.101683
	λ=0.5	0.087238	0.017448	0.757154	0.225398	1.591095	0.030675	0.201160	0.050290	0.415924
	λ=0.9	0.003572	0.000079	0.983869	0.016052	2.431603	0.104457	0.642024	0.160506	0.563448
0.8	λ=0.1	0.275784	0.496412	0.165502	0.338086	0.249914	0.000217	0.001068	0.000267	0.102987
	λ=0.5	0.147906	0.029581	0.588272	0.382146	1.292856	0.021474	0.101134	0.025284	0.499691
	λ=0.9	0.010364	0.000230	0.953198	0.046572	2.035911	0.082419	0.318847	0.079712	0.837101
0.5	θ=0.1	0.027289	0.001170	0.828590	0.170240	2.055226	0.063873	0.409329	0.102332	0.499368
	θ=0.5	0.024843	0.005323	0.953258	0.041418	2.032634	0.062851	0.395034	0.098759	0.509523
	θ=0.9	0.022027	0.008496	0.967777	0.023727	2.030115	0.062734	0.393880	0.098470	0.510263
0.8	θ=0.1	0.054238	0.002324	0.659315	0.338361	1.771908	0.050839	0.247847	0.061962	0.637343
	θ=0.5	0.059867	0.012829	0.887360	0.099811	1.620977	0.043888	0.164894	0.041223	0.701415
	θ=0.9	0.055291	0.021327	0.919114	0.059560	1.599550	0.042878	0.157440	0.039360	0.705622
0.5	ξ=0.1	0.010839	0.000929	0.960914	0.038157	3.298105	0.112156	0.166814	0.041704	0.562202
	ξ=0.5	0.022831	0.001957	0.917994	0.080049	2.244717	0.071654	0.347526	0.086882	0.520421
	ξ=0.9	0.028136	0.002412	0.899242	0.098346	1.996206	0.061343	0.411308	0.102827	0.501969
0.8	ξ=0.1	0.044842	0.003844	0.838299	0.157858	2.122984	0.065259	0.063865	0.015966	0.745414
	ξ=0.5	0.055802	0.004783	0.799571	0.195646	1.778659	0.050949	0.164696	0.041174	0.693349
	ξ=0.9	0.060464	0.005183	0.783470	0.211348	1.666280	0.046032	0.207048	0.051762	0.671379
0.5	N=10	0.031650	0.002713	0.886978	0.110309	1.917891	0.115060	0.316934	0.079234	0.489764
	N=15	0.028654	0.002456	0.897440	0.100104	1.997655	0.081648	0.370527	0.092632	0.500171
	N=20	0.027059	0.002319	0.903033	0.094648	2.041652	0.063262	0.400458	0.100115	0.505713
0.8	N=10	0.065881	0.005647	0.764735	0.229618	1.601668	0.085258	0.158758	0.039690	0.645840
	N=15	0.061781	0.005295	0.778870	0.215835	1.657188	0.060585	0.184735	0.046184	0.665150
	N=20	0.059533	0.005103	0.786662	0.208235	1.687761	0.046982	0.199153	0.049788	0.675757
0.5	ν=0.1	0.024500	0.002100	0.906255	0.091645	2.082385	0.065077	0.427041	0.106760	0.489745
	ν=0.2	0.025226	0.002162	0.905272	0.092566	2.067780	0.064436	0.416957	0.104239	0.495418
	ν=0.4	0.027059	0.002319	0.903033	0.094648	2.041652	0.063262	0.400458	0.100115	0.505713
0.8	ν=0.1	0.053611	0.004595	0.794865	0.200540	1.772817	0.050793	0.252996	0.063249	0.644888
	ν=0.2	0.055315	0.004741	0.792287	0.202972	1.742296	0.049446	0.232458	0.058114	0.655902
	ν=0.4	0.059533	0.005103	0.786662	0.208235	1.687761	0.046982	0.199153	0.049788	0.675757
0.5	β=0.1	0.000635	0.000054	0.997556	0.002390	2.925442	0.101261	1.060969	0.265242	0.098287
	β=0.5	0.018764	0.001608	0.931617	0.066774	2.183851	0.069648	0.495073	0.123768	0.437477
	β=0.9	0.058691	0.005031	0.801166	0.193804	1.690370	0.047125	0.198898	0.049725	0.669647
0.8	β=0.1	0.001696	0.000145	0.993475	0.006380	2.794019	0.095887	0.954594	0.238648	0.155034
	β=0.5	0.043524	0.003731	0.841387	0.154882	1.834685	0.053800	0.275448	0.068862	0.607394
	β=0.9	0.111738	0.009578	0.621452	0.368970	1.364368	0.031739	0.079988	0.019997	0.797028

Table 6: *Effect of μ, λ, θ, ξ, N, ν, and β on performance measures for SWV models.*

	MWV c=2	SWV c=2	MWV c=4	SWV c=4
$E(L_q)$	0.335060	0.334586	0.053407	0.053365
B_r	0.063306	0.063262	0.044173	0.044150
R_{ren}	0.401005	0.400458	0.131456	0.131353
R_{ret}	0.100251	0.100115	0.032864	0.032838
S_r	0.505335	0.505713	0.870924	0.870973

Table 7: *Effect of c on performance measures for SWV and MWV models.*

5. Following Tables 5-7, we observe that

$$\begin{aligned}
&\pi_{0,0}(\text{single working vacation}) &&< \pi_{0,0}(\text{multiple working vacation}),\\
&E(L_q)(\text{single working vacation}) &&< E(L_q)(\text{multiple working vacation}),\\
&E(L)(\text{single working vacation}) &&< E(L)(\text{multiple working vacation}),\\
&P_{wv}(\text{single working vacation}) &&< P_{wv}(\text{multiple working vacation}),\\
&B_r(\text{single working vacation}) &&< B_r(\text{multiple working vacation}),\\
&R_{ren}(\text{single working vacation}) &&< R_{ren}(\text{multiple working vacation}).
\end{aligned}$$

While

$$\begin{aligned}
&P_b(\text{single working vacation}) &&> P_b(\text{multiple working vacation}),\\
&R_{ret}(\text{single working vacation}) &&> R_{ret}(\text{multiple working vacation}),\\
&S_r(\text{single working vacation}) &&> S_r(\text{multiple working vacation}).
\end{aligned}$$

Then, we conclude that single working vacation model has better performance measures than multiple working vacations model. The obtained results match with our expected intuition.

7. Conclusion

In this research work, we considered a finite-buffer discrete-time multiserver queueing system with Bernoulli feedback, single and multiple working vacation policies, balking, reneging in busy

and working vacation periods, and retention of reneged customers, under late arrival system with delayed access (LASDA). The practical application of the proposed queueing system can be found in many real world situations, such as call centers, manufacturing and production systems, post offices, etc. The closed-form expressions for the steady-state probabilities of the system size were derived using the recursive method. Useful performance measures were obtained. Then, we developed cost model and performed a convenient optimization using a quadratic fit search method (QFSM) in order to get the optimum values of the service rate during busy period for different values of service rate during working vacation period, vacation rate, impatience rate and number of servers. Further, important numerical results were illustrated showing the applicability of the theoretical study.
For further work, it seems to be interesting to extend the considered model to more complex queues such as discrete-time $Geo^X/Geo^X/c$ feedback queueing model with single and multiple working vacations, impatient customers, service breakdowns and repairs.

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