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'MIXING APPLES AND ORANGES': NOTE ON A BEWILDERING APPLICATION OF EULER'S THEOREM TO THE S-SHAPED TOTAL PRODUCT AND RELATED CURVES

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ABSTRACT

In this research note, the Sigmoid-shaped total product (TP) curve and some extended versions (previously known as the symmetry) of the three stages of production are taken as subjects for critical discussion. We make the following arguments: (1) The extended version, which plots all four of AP_L, MP_L, AP_K and MP_K as inverted U-shaped curves on the same coordinate plane utilizing Euler's theorem to narrate and prove that the four curves can generate perfectly symmetric three stages, constitutes not only illogical drawing but also may involve improper reasoning that transgresses the boundary between disjoint sets. (2) When Euler's theorem is applied precisely to some of its accredited objects of TP curves, of which the Cobb-Douglas function often serves as the standard representation, the distinction among the alleged simple three stages of production would be inherently nonexistent in the first instance, thus dissolving the extended three-stage version accordingly. (3) If our potentially unprecedented theoretical elaboration and deduction in the penultimate section are validated and accepted, it appears there is no possibility for another proposed extended version, of which K/L has been directly assigned as an argument, to bear fruit in the end.

1. INTRODUCTION

As a long-standing practice, most introductory economics textbooks spend at least a section on introducing the Sigmoid-shaped total product curve, and whenever their authors choose the coordinate plane to illustrate the production function(s)—hereafter referred to in its common form $Q=f(L,K)$[1]—behind the curve, it is always accompanied with graph of average product (AP) and marginal product (MP) curves of the variable factor—L for most of the cases. The use of such-shaped TP curve probably comes from the logistic growth curve discussed in the demography literature, with the presence of an *inflection point* on the curve being a key characteristic.[2] While these textbook authors may have plotted, in their earlier chapter on (total) cardinal utility, something resembling an S-shaped curve, but the long-established concept of *law of variable proportions*[3] was never employed to illustrate the relative trend between average and marginal utilities; instead, it's always used to interpret the relationship between the variable factor's AP_L and MP_L, known as the three stages of production.

Given the S-shaped TP curve's fair share of academic and educational attention, further expounding might seem unwarranted; nevertheless, this paper will investigate some *extended version* (previously and more commonly known as the *symmetry*) of the three stages of production that arises through extrapolation. It will then review and narrate how a careless application of Euler's theorem may result in inadequate reasoning and erroneous inferences. But where did our distinctive problematic originate? When the first co-author (of the three of us) started, as a newly appointed university teacher, searching for reliable economics textbooks, it's surprising to discover that the graph of symmetric three-stage production and related wordings had been absent for more than a decade *without any explanation*. Initially, he speculated if it might be due to that graph's somewhat complex structure—for undergraduates at least; nevertheless, inspired then by revisiting and rereading Friedman (1962) and Hibdon (1969), the first and the third co-authors came to identify the most likely culprit: the application of Euler's theorem to a wrong object. After considerable effort and reflection, we recruited a colleague, now listed as the second co-author, to refine and develop an initially vague idea—TP^ in our fourth section—and, fortunately, we have been able to present it for further scrutiny.

While we wish and ask the potential readers to follow the below paragraphs and sections patiently, the following statements—criticism raised and explanations provided by our endeavors—should be kept in mind from the start: (1) In said extended version, the (smoothly) inverted U-shaped curves of AP_K and MP_K construed by taking K (or actually K/L if considering variable proportions) as an independent variable should not have been placed alongside the anterior and given AP_L and MP_L curves on the same coordinate plane, let alone compared locations against one another. (2) The S-shaped TP curve with its accompanying inverted U-shaped AP and MP curves *as a unity*—the focal point in this paper's figures and functions—cannot be derived from any homogeneous function, making it inappropriate here to apply Euler's theorem and certain formulas derived from it to explain and prove the symmetric three stages. (3) Rigorous and flawless applications of Euler's theorem (and its derivatives)—which work only on homogeneous functions—divulge how the prospective star feature AP and MP curves would not be inverted U-shaped; in one of such cases, the Cobb-Douglas production function as the usual and standard example, not only is the distinction among the three stages of production nonexistent from the very start, but also there would not be any extended three-stage version to serve as explanandum.

Although it seems redundant to state that the literature covered in this paper focuses on textbooks (and treatises) in hopes of reaching instructors and students in the field of economics, the second section would still, of course, start with the three-stage diagram with which our readers must have been familiar backwards and forwards, and then bring in the role of Euler's theorem for critical discussions in this section and also its succeeding one. Perhaps against all the odds, the third section will impart the lesson that, no matter how captivating the reasons for far-fetched application of Euler's theorem to the S-shaped TP and inverted U-shaped AP and MP curves might be, the theorem has nothing to do with the latter as a whole. Then, in the penultimate section, we offer a *potentially unprecedented* theoretical (re)formulation and conjecture as to which omission might have created another extended version, of which K/L has been directly assigned as an (additional) argument, of the three stages of production; nevertheless, we confess being agnostic about what the future may hold for our theoretical formulation in the end.

2. THE EXTENDED THREE-STAGE VERSION(S) EXTRAPOLATED FROM THE SIMPLE ONE

2.1. S-shaped and inverted U-shaped curves

Even without specifying which mathematical formula they describe, figure 1a and 1b are representations of an S-shaped and two inverted U-shaped curves mentioned above.[4] In figure 1b, the MP_L curve with no inflection point on it first goes up before bending down, and then crosses AP_L—again, without inflection point—at the latter's vertex, in which case the three stages of production are:

(1) The first stage: AP increases (while MP curve has passed through its own vertex).

(2) The second stage: AP decreases while MP has not yet reached 0.

(3) The third stage: MP moves from 0 into the negative.

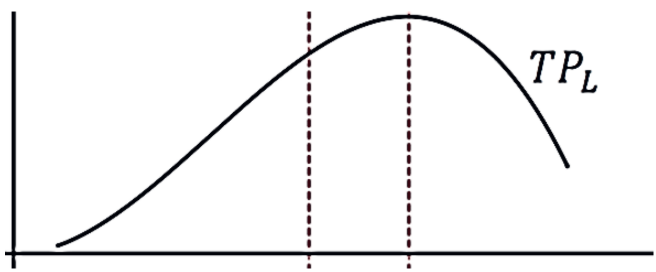

Figure 1a. S-shaped TP curve

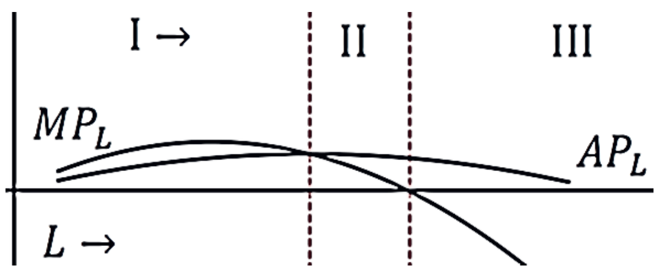

Figure 1b. The three stages of production

Before elaborating on certain extended version[5]—which for now we refer to AP and MP curves on the same coordinate plane having two lines each (four in total)—of the three stages of production, let us put forward several relevant but secondarily important points: (1) When discussing such a TP curve, most scholars would argue that it would be extremely 'unreasonable' for any production decision-maker to enter the third stage where *MP* falls into the negative value. After all, if a narrow assembly line (in this paper, *K* serves as such an example) is packed with so many laborers (*L*) that are clashing into each other's work progress, then that is a clear sign of harmful relative ratio of factors —*K/L* being too low. However, DeSerpa (1988, pp.183-184) and Shone (1981, p.112) have both raised interesting cases such as blue whale hunting and dairy farmers in some areas of the UK; therefore, it is necessary to acknowledge that the third stage does indeed correspond to some real-world production patterns.[6] (2) By contrast, in the first stage, which includes the prospects of increasing returns, it seems like initially a small number of personnel run wearily around the two ends of the assembly line; hence, increasing the variable factor's relative usage can boost the AP_L. (3) Although the second stage is generally accepted to be a reasonable production stage, sooner or later we must confront other economic and industrial factors such as cost and market structures. On this inviting topic, Tangri (1966, pp.485-489) and Seagraves & Pasour (1969) are indispensable references to begin with.

Next, let us return to the schematic figure 1a and 1b: given that the variable on the horizontal axis—*L*—is one of varied value, when productive factors fail to maintain constant proportion between them, we may take amounts of *L* on the horizontal axis as also representing *L/K*—the relative ratio mentioned in the previous paragraphs—especially when *K*=1.[7] Hence, as *L/K* increases moving right along the horizontal axis, we can say that *L/K* decreases as it moves left. In other words, in a *very loosely defined* sense, if we can establish, towards the 'somewhat distant' right side of the current horizontal axis, an 'origin point' based on *L* approaching infinity—$K/L \to 0$—then as we move left from this (virtual?) origin, *K/L* naturally goes from low to high. It is exactly here that, in addition to the given AP_L and MP_L curves, we expect the rise of an extended version of the three stages of production—which includes the stage where $AP_{K/L}$ is on the rise, the stage where $AP_{K/L}$ moves downward while $MP_{K/L}$ remaining positive, and the stage where $MP_{K/L}$ finally enters the negative—that attempts to treat *K/L* as an argument/variable as seen in figure 1c.

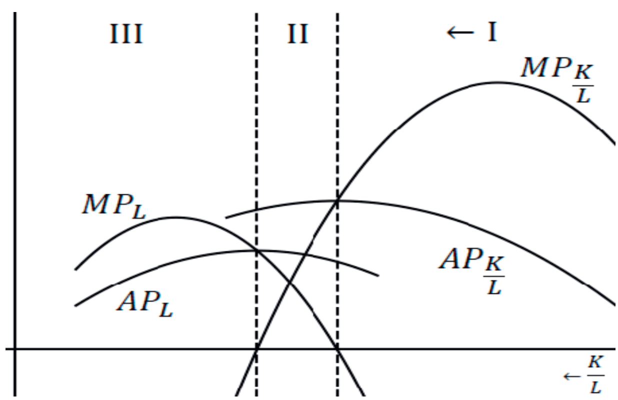

Figure 1c. The extended three stages of production

Take a closer look at figure 1c. To make sense of the direction of curves $MP_{K/L}$ and $AP_{K/L}$ and identify the possible division between the three stages with respect to L and K/L, the figure has to be read also from right to left, and we acknowledge that the following two near-perfect characteristics to be—setting aside accuracy for now, with further discussion coming in the next sections—products of deliberate design: (1) Directly below where $MP_{K/L}$ and $AP_{K/L}$ curves intersect lies MP_L=0, which is where MP_L curve intersects the horizontal axis; (2) Directly above the point on the horizontal axis where $MP_{K/L}$=0, we find the point of intersection between MP_L and AP_L curves. In this way, the first (or third) stage of MP_L and AP_L corresponds to the third (or first) stage of $AP_{K/L}$ and $MP_{K/L}$; in other words, they bear symmetry. The corresponding relation as seen on the right side of figure 1c is widely interpreted as such: relative to K, too much L creates a negative MP value while the symmetry exhibited on the figure's left suggests that a negative MP value results from relatively excessive K.

2.2. Then came the Euler's theorem

As to the possible role of Euler's theorem here for the S-shaped and inverted U-shaped curves as our subject of study, while setting aside the intriguing question of whether anyone can accurately assign coordinate values to each point on the $MP_{K/L}$ and $AP_{K/L}$ curves,[8] Mundlak (1958, pp.758-760) was probably, based on this study's research to date, the earliest piece of literature to introduce Euler's theorem to elaborate and justify (*his own* table of) 'classification of stages of production' in which homogeneous functions are *not* restricted to the first order. In terms of treatises or textbooks, Danø (1966, p.206) was probably the first to employ Euler's theorem (and homogeneous functions of degree one) explicitly to explain something *comparable to* the relative positions and points of intersection among the four curves in our figure 1c. Similar and absolutely important works after Danø include Bilas (1971, pp.132-138, particularly figure 6-11-b), Intriligator (2002[1971], pp.186-189), Ching & Yanagida (1985, pp.109-110), Chacholiades (1986, pp.582-583), Doll & Orazem (1992, pp.206-209, particularly figure 6-6), and Wetzstein (2013, pp.267-268).[9]

According to Euler's theorem, all homogeneous functions of degree r—which we would continue to refer to as $Q=f(L,K)$—bear the following characteristic: $r{\cdot}Q=MP_L{\cdot}L+MP_K{\cdot}K$; so, when r=1,

$$Q=MP_L{\cdot}L+MP_K{\cdot}K. \quad (1)$$

Next, let us divide (1)—expressed fully as formula (1) from now on, and the same goes for the following (2) and (3)—by L, and then it can be seen that when K/L is not equal to 0,

$(Q/L\equiv)AP_L=MP_L+MP_K{\cdot}(K/L)$ implies:

$$MP_K=0\leftrightarrow AP_L=MP_L,\ MP_K>0\leftrightarrow AP_L>MP_L \text{ and}$$
$$MP_K<0\leftrightarrow AP_L<MP_L. \quad (2)$$

Similarly, if we instead divide formula (1) by *K*, it can still be understood to mean that when L/K is not equal to 0,

$$AP_K = MP_L \cdot (L/K) + MP_K \text{ implies:}$$
$$MP_L = 0 \Leftrightarrow AP_K = MP_K,\ MP_L > 0 \Leftrightarrow AP_K > MP_K \text{ and}$$
$$MP_L < 0 \Leftrightarrow AP_K < MP_K. \quad (3)$$

After all of these, let us inspect formula (3) firstly. Although its foundation is just an arithmetical division operation between *Q* and *K*, it is mostly interpreted, without showing the exact mathematical function form, as follows: (*if both curves are inverted U-shaped,*) the point of intersection of AP_K and MP_K curves corresponds to MP_L=0; when MP_L is positive, it corresponds to the interval where AP_K is decreasing; when MP_L is negative, it corresponds to the interval where AP_K is still increasing. Similarly, formula (2) is also interpreted as: (*if both curves are inverted U-shaped,*)[10] the point of intersection between AP_L and MP_L corresponds to MP_K=0; when MP_K is positive, it corresponds to the interval where AP_L is decreasing; when MP_K is negative, it corresponds to the interval where AP_L is still increasing.

Looking back at figure 1b and 1c, it's as if here comes something quite naturally to be regarded as another extended version. How does it compare to what this paper offered in previous paragraphs for elaborating (perhaps somehow unrigorously!) figure 1c with its horizontal axis representing *L* and/or *K*/*L*? In fact, if there were better reason (especially one backed by mathematical basis and not just empirical and/or intuitive impressions) that faultlessly justifies replacing the figure's inverted U-shaped $AP_{K/L}$ and $MP_{K/L}$ with AP_K and MP_K curves, followed sequentially by a satisfactory explanation of how symmetric three stages are indeed plausible, then one can imagine that is where we should *not* embark on critique and rebuttal of such kind of version that dauntlessly applies Euler's theorem (and its derived formulas). But of course we should and would try immediately; so, in the upcoming main section and remaining part of this one, we would conceptually replace the horizontal axis variables of figure 1c as *L* and/or *K*.

First of all, the existing TP_L curve in figure 1a already represents a *short run* production function; therefore, any point on this curve presupposes a constant amount of the fixed factor—which refers to only one assembly line (*K*=1) in our case—and because we are *not* coping with moving along any specific isoquant, simultaneous changes in the amounts of both *L* and *K* along the same TP_L curve will be illogical and better not be allowed. With or without focusing on an S-shaped TP curve with diverse shapes of AP and MP, this also means that if we take a TP curve with a logically preset fixed constant value of *K* and apply partial differentiation to obtain an MP_L curve, it would be impossible and illogical to continue using that fixed-value *K* to find the MP_K curves, let alone introduce any MP_K curve to the same coordinate plane and compare its position against that of an anterior MP_L curve.[11] In other words, unless we are willing (or forced!) to accept extraordinarily relaxed, compromised requirements, the extended three-stage version illustrated in figure 1c immediately violates logical and mathematical rules in plotting and doing partial differentiation.

Then, let's take a different perspective for examining the problems that arise from juxtaposing MP_L *and* MP_K curves by considering a cubic polynomial—$Q=-L^3-K^3+a_1a_2L^2+a_1a_2K^2+2a_1a_2LK$—taken from Puu (1989, p.122), which *can* generate an S-shaped TP curve with inverted U-shaped curves of AP and MP. Let $K=1$; here, the MP_L function is $-3L^2+2a_1a_2L+2a_1a_2$ and has been plotted in figure 1b. Next, let's add an MP_K curve so as trying to expand figure 1b into figure 1c in preparation for the final completion of the extended three-stage version. However, after doing the partial derivative with respect to K, we get $\partial Q/\partial K=-3K^2+2a_1a_2K+2a_1a_2L$; so, which MP_K curve shall be the one plotted onto the same coordinate plane? As L varies in value, *it is impossible to obtain a unique* MP_K *curve* here; and we are sorry to declare that, for those who regard formula (1), (2) and (3) as nothing but constructing and plotting the four functions/curves precisely, this alone should count as a shocking or even fatal blow,[12] no matter whether our subsequent discussion being right or wrong and whether the subject being an S-shape with bell-shaped AP and MP curves or not!

But *even if* we set aside (for the moment) the questions raised in the above two paragraphs, can formula (1) really be applied to this cubic polynomial (and any one of its S-shaped curves)? If so, then performing subsequent calculations and plotting based on this polynomial should help us obtain (partial) confirmation of the extended version. However, a simple differentiation and summation can suffice as a counterexample: $MP_L{\cdot}L+MP_K{\cdot}K=(-3L^2+2a_1a_2L+2a_1a_2K){\cdot}L+(-3K^2+2a_1a_2K+2a_1a_2L){\cdot}K=-3L^3-3K^3+2a_1a_2L^2+2a_1a_2K^2+4a_1a_2LK$, which means $MP_L{\cdot}L+MP_K{\cdot}K\neq Q$. Since this polynomial as such does not seem to bear the characteristic of formula (1) inherently, it is much more unlikely to have the characteristics of formula (2) and (3), which are derived from formula (1). Now let us immediately assert boldly the following: although the above polynomial is but a *single* example, it might be enough to show—even tentatively—that almost all of the function(s) behind any S-shaped TP curve (with its inverted U-shaped curves of AP and MP) would be non-homogeneous. If this indeed is the case, then we should refrain from arbitrarily applying formula (1) and its derivatives on said TP curves because formula (1) is characteristically exclusive to homogeneous functions of degree one.

Besides, with just enough attention paid to the graph (of the usual and standard homogeneous function of degree one, the Cobb-Douglas in our case, as we soon see), the reasons why homogeneous functions can hardly yield the inverted U-shaped AP and MP curves might become a little clearer—though we insist on discussing this point in detail in the ensuing section. If so, then why is it that, in this extended version, almost nobody seems to have ever questioned what qualifies Euler's theorem to serve as a pivotal means for inference? In our opinion, those who believe that formula (2) and (3) can sufficiently explain and justify figure 1c, either might have chosen to ignore the idea that the U-shaped curves there and homogeneous functions are better represented as belonging to disjoint sets,[13] or might unobtrusively believe that: even though *homogeneous functions might be unable to generate inverted U-shaped AP and MP curves*, by tentatively and boldly *assuming* that certain non-homogeneous functions might behave like what homogeneous functions will do and then conveniently employing the results derived from formula (1), (2) and (3), the outcome (seemingly) happens to explain the symmetric three stages found in figure 1c. In other words, they seem to believe that 'a good end always justifies the means!' By contrast, we insist in all earnestness that such 'mixing apples and oranges' confusion among function types has better be resolved in advance and that the possible deficiencies in the contents of formula (2) and (3) might be the true culprits of the 'ostensibly good end.' Hence, in the next section—using mathematical examples with or without graphical aids—we would continue to elaborate and try to convince our readers that the symmetric three-stage of figure 1c is close to mathematical fiction, even if a handful of homogeneous functions of degree one actually might generate a *simple* version of three stages of production.

3. TRANSGRESSING THE BOUNDARY BETWEEN DISJOINT SETS HERE HAS DONE NOTEWORTHY BUT NEGLECTED HARM

3.1. 'Assuming first-order homogeneity' will not always lead to an S-shape

In some detail-oriented publications, Euler's theorem constantly appears with longer, more complete terms such as 'Euler's theorem for/on homogeneous functions'—as seen in Petri (2021, p.168), Pemberton & Rau (2016, p.284), and the most relevant Creedy (2016, p.311). This should have served as a reminder that the theorem is irrelevant to non-homogeneous functions, therefore not even indirectly applicable to the latter.[14] Moreover, among the existing literature on models of two-factors-and-one-product, every explicit example of a mathematical function behind an S-shaped TP_L curve with its accompanying inverted U-shaped AP and MP curves is non-homogeneous. In addition to the previously mentioned Puu (1989), there is an interesting example recently given by Moss (2022, pp.5-8) that highlights a certain 'Zellner's production function'— $f(L,K)=(0.0005433)L^3/\{-1+\exp[(0.01794)(L/K)]\}$— which shows an S-shaped TP curve and seemingly (partial) bell-shaped curves of MP and AP; but, unlike the Sato/Rowe type mentioned in our endnote 2, it's equipped with sort of 'well-behaved' isoquants, and some differentiation and multiplication would be enough to reveal that it most certainly is not a homogeneous function.[15]

As mentioned before, formula (1) shows characteristics that are unique to any homogeneous function of degree one, and formula (2) and (3), which are obtained by simple division operation, also seem to represent characteristics that are *exclusive* to said function (please allow for revisions on this matter later). In this case, given the premise that S-shaped TP with their inverted U-shaped AP and MP curves as a unity cannot, indeed, come from homogeneous functions—a point we hope would be accepted—if someone continues to employ formula (2) and (3) to explain and justify the claim that the four curves in our figure 1c must necessarily have perfect locations and intersections, then that would actually be a practice of mistakenly placing non-homogeneous functions into the somehow downgraded realm of homogeneous functions—whether said person is aware of this or not.

Taking a step back, even though we argue strongly—with endnote 15 to back us up further—that S-shaped and inverted U-shaped curves in figure 1a and 1b must come from non-homogeneous functions, given that both lack generalized forms and the possibility for at least one (kind of) homogeneous function being able to generate S-shaped TP_L curve with (partial) bell-shaped curves of AP and MP, this paper, at this point, seems to be shy of adequate persuasiveness. Nonetheless, we still advise those who believe that, by applying Euler's theorem for homogeneous functions and division operations on it they can derive figure 1c from the given figure 1a, to take an additional question here seriously—*Can a homogeneous function of degree one like the Cobb-Douglas be used to plot an S-shaped TP curve?* If this standard first-order homogeneous function answers 'NO' to the question, then it means that here even the simple three stages represented by figure 1b would not exist either, because now there would be no inverted U-shaped AP and MP curves in the first place. In other words, '*assuming first-order homogeneity*' *alone* will not always lead to an S-shape. Since figures cannot speak for themselves but can be misread at the first chance, we offer a thought experiment that, for the time being, does not rely on figures; and this would help us gradually explain how and why we should not recklessly indulge the superficial appeal of mathematical theorems to the point we compromise our reasoning and judgment abilities.

Imagine a scenario in which there is a sudden blackout during class that prevents the instructor from being able to take any laid-out homogeneous function, run it online through GeoGebra, and display its figures on the electronic screen in real time. However, where there's a will, there's a way—we still have our pen and paper to perform tasks like calculating derivatives or making qualitative judgments, which unexpectedly allows us to temporarily break free from (excessive?) reliance on graphical intuition. Next, even if there doesn't seem to be any generalized formula for *the* homogeneous functions, let us boldly adopt the usual and standard Cobb-Douglas function—$Q=L^{\alpha}K^{1-\alpha}$, $1>\alpha>0$—to provide some (perhaps somewhat redundant) calculations and analysis.

Let K=1 (or L=1), and promptly obtain the current MP_L and AP_L functions (or MP_K and AP_K for that matter—we would no longer repeat writing and calculation of this sort below) before solving for their own first and second derivatives with respect to L. Even without graphical representation, we immediately discover that

$MP_L=\alpha/L^{1-\alpha}$ (>0), $AP_L=1/L^{1-\alpha}$ (>0);

$\partial MP_L/\partial L=\alpha(\alpha-1)/L^{2-\alpha}$ (<0), $\partial AP_L/\partial L=(\alpha-1)/L^{2-\alpha}$ (<0);

$\partial(MP_L/\partial L)/\partial L=\alpha(\alpha-1)(\alpha-2)/L^{3-\alpha}$ (>0), and

$\partial(AP_L/\partial L)/\partial L=(\alpha-1)(\alpha-2)/L^{3-\alpha}$ (>0).

In other words, at this point, AP_L and MP_L are not only both positive, but their slopes are negative throughout. In addition to not being inverted U-shaped at all, the two curves have absolute values of slopes decreasing all the way as they approach the lower right of the horizontal axis *without ever* intersecting—features that graphs would clearly demonstrate on screen as the power comes back in class (see figure 2a and 2b with K=1 and α=0.5). All this stems from the fact that the TP curve

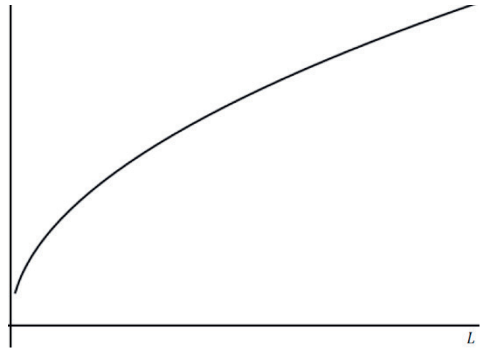

Figure 2a. TPL curve of $Q=L^{0.5}K^{0.5}$

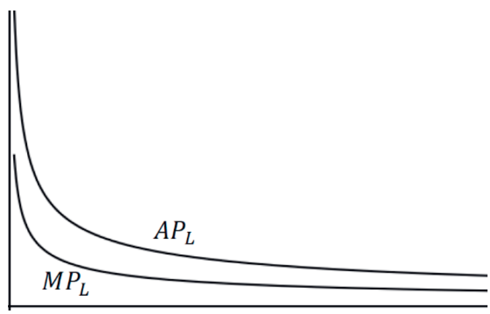

Figure 2b. AP_L and MP_L curves of $Q=L^{0.5}K^{0.5}$

at this time would never exhibit an S-shape, which means there would be no inflection point on the curve; hence, there obviously doesn't exist some simple three stages of production (figure 1b) here, *let alone any extended three-stage figure* (figure 1c). Without the extended version, formula (2) and (3), originally considered *explanantia*, seem to become useless and relegated to an important role of mere computational purposes.[16]

Upon regaining the intuitive support provided by graphics, let us immediately raise two (perhaps already repeated) points to summarize the reasoning and conclusions from the previous main section up until now: (1) As the focal point of this paper's figures and functions, the S-shaped TP curve with its inverted U-shaped curves of AP and MP as a whole do not come from homogeneous functions; hence, formula (1), (2), and (3) shouldn't be used to explain the extended version. (2) Even if someone initially and consciously set the production function as an indubitably accredited first-order homogeneous function, after rigorous reasoning supplemented by graphical aid, formula (2) and (3) might then be rendered useless if the TP function that is the true star of the show here cannot generate inverted U-shaped AP and MP curves, and then there naturally wouldn't be any extended three-stage figure to serve as explanandum; after all, even the simple version went out the window one step earlier. However, if this is the case, wouldn't it render all relevant scholarly endeavors that made use of formula (2) and (3) in their literature efforts almost in vain? The answer, we're afraid, is 'perhaps YES;' for all intents and purposes, setting up instead a Sato/Rowe type of first-order homogeneous function with partial bell-shaped AP and MP curves and acknowledging its unusual shape of isoquants with unexpected number of ridge lines—again, see Truett and Truett (2006)—have seemed unable to take the place of the Cobb-Douglas. To this point, we would propose a belated but emphasis-worthy *revision*—at least for the Cobb-Douglas—even if so far nobody has questioned whether the contents of formula (2) and (3) are one hundred percent accurate and reasonable for each and every homogeneous function of degree one.

3.2. Setting the record straight and righting the possible wrongs

We have mentioned several times already that formula (2) and (3) were initially just arithmetical operations of dividing formula (1) by either L or K, and the interpretations of these two derivatives seem flawless and logically sound. But in fact, the arguments presented so far in this main section—along with graphical aid from figure 2b—are enough to suggest that: while formula (2) and (3) may seem, on surface, to be nothing more than simple calculations, they actually have somewhat distorted a fundamental characteristic of the Cobb-Douglas, i.e., that MP curves of this function do not intersect the horizontal axis in the first quadrant and MP is consequently always positive. Hence, if formula (2) and (3) are to represent said function exactly and flawlessly, they must each rid their contents of the two statements regarding MP being non-positive values. Apparently, if previous attention was directed only at whether the division operation itself was performed for each and every first-order homogeneous function or not, then the necessity of such riddance wouldn't easily come to mind.

So, is it possible that some scholars may actually be aware—after all, teachers and researchers inevitably use computers or blackboards to sketch graphs—that formula (2) and (3) are in need of correction at least for the Cobb-Douglas, but have 'obliviously' transposed formula (2) and (3) from the realm of homogeneous functions of degree one into that of non-homogeneous functions simply because the inverted U-shaped MP curves in figure 1c are all drawn as intersecting the horizontal axis, meaning their values turn from positive to 0 and negative? As far as the transposition result is concerned, it is nearly tantamount to 'two lefts making a right.' In other words, they might have taken the inappropriate-as-it-is contents of formula (2) and (3) and applied directly to those non-homogeneous functions that have nothing to do with Euler's theorem for homogeneous functions, yet it *seems*—surprisingly—extremely reasonable; after all, don't we all see how each of their *three* statements inadvertently and ostensibly correspond to the pattern of the *extended three*-stage figure illustrated by the four curves?

This superficially perfect accidental match might not only undermine the harm of realm transposition but also deepen the impression that 'a good end always justifies the means.' Or would it be, perhaps, that using formula (2) and (3) to explain the relative positions and intersections among the curves in figure 1c constitutes a form of *Texas sharpshooter fallacy*? Simply put, was it only because of the preconceived belief that the extended version is plausible and compelling that formula (2) and (3) were deemed perfectly suitable to be justification tools for narration and explanation? Since it is impossible for us to gain insight to every author's motive and context when they make their argument, we can only set aside the above questioning for now. Nevertheless, we should promptly come back and attend to the necessary correction to formula (2) and (3) to ensure nobody distorts the characteristics of the Cobb-Douglas—the usual and standard representation of homogeneous functions of degree one.

So, *how do we correct this*? Will the corrected (new) formulas be logically sound and suitable for explaining figure 1c? Let us still start with formula (3). As mentioned above, since MP_L here is never 0 or negative *for the Cobb-Douglas*, the correct statement to be deduced from formula (1) now should be written as follows:

From Euler's theorem for homogeneous functions of degree one,
$AP_K = MP_L \cdot (L/K) + MP_K$ implies that
$MP_L \neq 0 \leftrightarrow AP_K \neq MP_K$ and $MP_L > 0 \leftrightarrow AP_K > MP_K$. (4)

And this, or course, once more means that the MP_K curve does not intersect with the AP_K curve—here (and below, *mutatis mutandis*), the vital question of whether these two curves can be derived on the basis of the anterior MP_L and AP_L curves is again left aside! Likewise, formula (2) must be corrected and written in full as follows:

From Euler's theorem for homogeneous functions of degree one,
$AP_L = MP_L + MP_K \cdot (K/L)$ implies that
$MP_K \neq 0 \leftrightarrow AP_L \neq MP_L$ and $MP_K > 0 \leftrightarrow AP_L > MP_L$. (5)

This also means that the MP_L and AP_L curves do not intersect![17] After executing these two corrections, it becomes even more clear that any explanation for the extended three-stage version in our figure 1c still cannot rely on such kind of corrections; after all, in this figure, every set of inverted U-shaped curves of AP and MP intersect with each other, not to mention the latter passes through the former's vertex.[18]

Hence, it cannot be overemphasized that plotting all four of AP_L, MP_L, AP_K and MP_K as inverted U-shaped curves on the same coordinate plane and then utilize Euler's theorem for homogeneous functions as *the sole* basis for proving that the four curves can generate perfectly symmetric three stages, not only constitutes illogical drawing and improper reasoning that could transgress the boundary between disjoint sets, but also might misjudge what correct contents that said theorem's derivatives have. It's no wonder that, in the past ten or more years, the extended three-stage has been completely absent in the newer textbooks in the UK and US, and we are not sure if that is also attributed to the widespread popularity of drawing software, causing authors to realize that it is impossible to *simultaneously and exactly* draw all four inverted U-shaped curves in figure 1c by way of programming and running gratifyingly any homogeneous function, not even to mention non-homogeneous ones. Nevertheless, it's hoped that our critical analyses so far don't come off as 'beating a dead horse.'

4. ANOTHER EXTENDED VERSION FOR THE WRONG REASON

Then, let's take stock of a relevant issue that has been delayed until now. In contrast to the approach of utilizing Euler's theorem for homogeneous functions, there is another extended version we mentioned quite early on that might have come from taking a production function of no specific form in advance and transforming it by means of unexpected mathematical analysis and improper judgment; that is, rewriting $Q=f(L, K)$ as if K/L can be an argument *from which a three-stage version always emerges without a hitch*. While it may be only our preliminary speculation, if we peruse meticulously the early works of Friedman and Hibdon, they could be interpreted as having implicitly and informally performed such transformations. To show how this version might work at first glance but fails to bear fruit in the end, the following are several unprecedented elaborating steps from (A1) advancing to (D)—starting with redefining the production function, to then calculating the first derivative of AP_L function itself and also deriving the MP_L function, and finally offering an account for this: under what circumstances of omission is there a next to impossible chance for the extended version that treats K/L as an independent variable to be established?

(A1) As usual, $TP=f(L,K)$, $AP_L=TP/L$ and $MP_L= \partial TP/\partial L$. Here, we represent them in slightly more complex forms as:

$TP(L,K)$, $AP_L(L,K)$ and $MP_L(L,K)$.

(A2) Define *deliberately* all the following—$TP^{\wedge}$, $AP^{\wedge}$ and $MP^{\wedge}$—as functions of $\frac{L}{K}$ (and/or L):

$$TP^{\wedge}(L,\frac{K}{L})=TP(L,K)/(\frac{L}{K})=[TP(L,K)\cdot(\frac{1}{L})]\cdot K$$

$$AP^{\wedge}_{K/L}(\frac{K}{L})=TP^{\wedge}(L,\frac{K}{L})/(\frac{K}{L})$$

$$MP^{\wedge}_{K/L}(\frac{K}{L})=\partial TP^{\wedge}(L,\frac{K}{L})/\partial(\frac{K}{L})$$

$$MP^{\wedge}_{L}(\frac{K}{L})=\partial TP^{\wedge}(L,\frac{K}{L})/\partial L_{(DE)}.$$[19]

Then come some properties as follows,

(1) $TP^{\wedge}(L,\frac{K}{L})/K=TP(L,K)/L$

(2) $TP^{\wedge}(L,\frac{K}{L})=AP_L(L,K)\cdot K$

(3) $AP^{\wedge}{}_{K/L}(\frac{K}{L})=TP^{\wedge}(L,\frac{K}{L})/(\frac{K}{L})=[TP(L,K)/(\frac{L}{K})]/(\frac{K}{L})=TP(L,K)$

(4) $AP^{\wedge}{}_{K/L}(\frac{K}{L})\cdot(\overline{L})=TP^{\wedge}(L,\frac{K}{L})/K.$

(B) Solve $\partial AP_L(L,K)/\partial L$:

$=\partial[TP(L,K)/L]/\partial L$

$=\partial[TP^{\wedge}(L,\frac{K}{L})/K]/\partial L$

$=(\frac{1}{K})\cdot\{[\partial TP^{\wedge}(L,\frac{K}{L})/\partial(\frac{K}{L})]\cdot[d(\frac{K}{L})/dL]+[\partial TP^{\wedge}(L,\frac{K}{L})/\partial L_{(DE)}]\cdot(dL/dL)\}$

$=(\frac{1}{K})\cdot[MP^{\wedge}{}_{K/L}(\frac{K}{L})\cdot(\frac{-K}{L^2})+MP^{\wedge}{}_{L}(\frac{K}{L})].$

(C) And find $MP_L(L,K)$:

$=\partial TP(L,K)/\partial L$

$=\partial[TP^{\wedge}(L,\frac{K}{L})\cdot(\frac{L}{K})]/\partial L$

$=[TP^{\wedge}(L,\frac{K}{L})\cdot d(\frac{L}{K})/dL]+$

$(\frac{L}{K})\cdot\{[\partial TP^{\wedge}(L,\frac{K}{L})/\partial(\frac{K}{L})]\cdot[d(\frac{K}{L})/dL]+$

$[\partial TP^{\wedge}(L,\frac{K}{L})/\partial L_{(DE)}]\cdot(dL/dL)\}$

$=[TP^{\wedge}(L,\frac{K}{L})\cdot(\frac{1}{K})]+$

$(\frac{L}{K})\cdot[MP^{\wedge}{}_{K/L}(\frac{K}{L})\cdot(\frac{-K}{L^2})+MP^{\wedge}{}_{L}(\frac{K}{L})]$

$=[TP^{\wedge}(L,\frac{K}{L})\cdot(\frac{L}{K})\cdot(\frac{1}{L})]+$

$(\frac{L}{K})\cdot[MP^{\wedge}{}_{K/L}(\frac{K}{L})\cdot(\frac{-K}{L^2})]+(\frac{L}{K})\cdot MP^{\wedge}{}_{L}(\frac{K}{L})$

$=[AP^{\wedge}{}_{K/L}(\frac{K}{L})\cdot(\frac{1}{L})]+$

$[(\frac{-1}{L})MP^{\wedge}{}_{K/L}(\frac{K}{L})]+(\frac{L}{K})\cdot MP^{\wedge}{}_{L}(\frac{K}{L})$

$=(\frac{1}{L})\,[AP^{\wedge}{}_{K/L}(\frac{K}{L})-MP^{\wedge}{}_{K/L}(\frac{K}{L})]+(\frac{L}{K})\cdot MP^{\wedge}{}_{L}(\frac{K}{L}).$

(D) Finally comes the conclusion (with implications not only for this section):

Conclusion D1: From (B), let $\partial AP_L(L,K)/\partial L=0$, we find that only when $MP^{\wedge}{}_{L}(K/L)$ is casually but deliberately ignored and eliminated will there be $[(1/K)\cdot MP^{\wedge}{}_{K/L}(K/L)\cdot(-K/L^2)]=0$. Compared to formula (2) from before, here it seems to imply that $\partial AP_L(L,K)/\partial L=0 \leftrightarrow MP^{\wedge}{}_{K/L}(K/L)=0$, which looks as if that the highest point on the

AP_L curve could correspond to the intersection between the $MP^{\wedge}_{K/L}$ curve and the horizontal axis!

Conclusion D2: From (C), let $MP_L(L,K)=0$, similarly we find that only when $MP^{\wedge}_L(K/L)$ is casually but deliberately ignored and eliminated will there be $(1/L)\,[AP^{\wedge}_{K/L}(K/L) - MP^{\wedge}_{K/L}(K/L)]=0$. Compared to formula (3) from before again, here it seems to imply that $MP_L(L,K)=0 \leftrightarrow AP^{\wedge}_{K/L}(K/L)=MP^{\wedge}_{K/L}(K/L)$, which looks as if that the highest point on the $AP^{\wedge}_{K/L}$ curve could correspond to the intersection between the MP_L curve and the horizontal axis!

In our opinion, $MP^{\wedge}_L(K/L)$ here of course *can be neither omitted nor assumed to be zero* in advance with no proper reason—it is part and parcel of steps (B) and (C) for calculating differentials—so this extended version of the three stages is implausible if not fictitious; and the true and ultimate conclusion of this penultimate section, naturally, has to remain the same as that of the previous one, which is: eventually, there is no sufficiently reliable reason for us to expand figure 1b into figure 1c.

5. BY WAY OF CONCLUDING: CASE CLOSED WITH CONFESSIONS

During the current theoretical journey, there have been some self-reflections and even minor debates among the three of us as co-authors. Since assuming homogeneity alone will not validate any extended three-stage version, might our narration be judged as aiming to 'destroy' Euler's theorem for homogeneous functions *per se*? And would other (two-factor) roadmaps adopting $TP^{\wedge}$ as the right generalized form of functions lead successfully to the promised land of economic analyses of production? We offer two '*conditional* NOs' as answers to get about and reach the end.

First of all, Euler's theorem works fine for any homogeneous function, though our subject of S-shaped and inverted U-shaped curves has nothing to do with the theorem as such; however, the *utterly static and descriptive* nature of Euler's theorem for homogeneous functions of degree one—formula (1)—could not be overemphasized. $Q=L^{0.5}K^{0.5}$ again as the example, it can be shown that there's only one meaningful MP curve, not two, even for trying to accomplish figure 1c with L and/or K as the variables on the abscissa; and, by a simple but inspirational mathematical exercise, let us quickly explain why and how.

(1) For $Q=L^{0.5}K^{0.5}$, $MP_L(L,K)=MP_L(K/L)=(0.5)(K/L)^{0.5}$ and $MP_K(L,K)=MP_K(L/K)=(0.5)(L/K)^{0.5}$; (2) suppose there are two points—'L=1 (; with K=4) → MP_L=1' and 'L=2 (; with K=4) → $MP_L=2^{-0.5}$'—on a MP_L curve logically derived from a given TP_L curve along which K's kept constant. Then (3), how to construct a seemingly acceptable MP_K curve to be situated, even if tentatively, in the same coordinate plane with the anterior MP_L curve? Here we have 'K=4 (; with L=1) → MP_K=1/4' and 'K=4 (; with L=2) → $MP_K=2^{0.5}/4$,' but they cannot be on the same MP_K curve logically derived from a TP_K curve with L kept constant. Fortunately, owing to 'K=2 (; with L=1) → $MP_K=2^{0.5}/4$,' we finally have

a *correlation*, i.e., 'K=4 (; with L=1) → MP_K=1/4' and 'K=2 (; with L=1) → $MP_K=2^{0.5}/4$,' to get the job done.

But we confess promptly that manufacturing and arranging *this pair* of MP_L and MP_K curves into figure 1c are never effective and appealing. Of course, it's still intriguing to know: whoever was the first to (turn L/K upside down and) regard K/L as an argument, moving from right to left on the abscissa of figure 1c, for explaining the law of variable proportions and/or the extended three-stage here? Have the scales ever fallen from her/his eyes that this argument *would not* be worth the candle for those non-homogeneous functions incompatible with formula (1)?

Since from the outset we have taken the instructors and students as potential readers, and the target audience for textbooks can vary widely, some additional advice seems appropriate: if introductory textbooks are determined to stick to using only discrete numbers to introduce S-shaped TP and inverted U-shaped AP and MP curves, then a simple three-stage version should suffice; moreover, the instructor should remind students early on that *these curves as a whole come from non-homogeneous functions*, which means they shouldn't reference Euler's theorem for homogeneous functions here, unless partial bell-shaped AP and MP curves being the protagonists in fact. If ever our readers find themselves in need of more advanced content—including continuous, differentiable, or multivariate mathematical functions—then it is hoped that the arguments presented thus far could convince our colleagues in the academic communities that: Euler's theorem for homogeneous functions is of absolutely no help for the exact topic discussed in this paper; in other words, this theorem cannot justify the extended three-stage version that figure 1c wants to depict.

Still, for our potential critics, there might be another serious problem in need of addressing; that is, why have we, in the previous section, deliberately defined $TP^{\wedge}(L,K/L)$ as $TP(L,K)\cdot(K/L)$ instead of another possible form—such as Mundlak (1958, p.757)? As mentioned twice at least, we were actually inspired and assisted by two sources as follows (expressed in terms of our notation): (1) Friedman's (1962, p.125) table 5, columns 5~7, which undisguisedly express $MP^{\wedge}_{K/L}(K/L)$; and (2) Hibdon's (1969, p. 120) table 5-1, *column* 4 and *column 12*, illustrating $AP^{\wedge}_{K/L}(K/L)=TP(L,K)$ even more clearly, which lead seamlessly and noteworthily back to $TP^{\wedge}(L,K/L)=TP(L,K)/(L/K)$! Even though they never used continuously differentiable functions in the corresponding texts of their works, it seems that the so-defined $TP^{\wedge}(L,K/L)$ is the true basis for completing their input-output schedules; besides, in an ancillary appendix (with only three pages)[20] we have also offered details about how to reformulate some TP curves used earlier—$Q=-L^3-K^3+a_1a_2L^2+a_1a_2K^2+2a_1a_2LK$ and $Q=L^{\alpha}K^{1-\alpha}$—into the proposed TP^ functions to make the previous section's steps (B) and (C) well executed.

But in the very end, with this note's case closed, we would like also to confess being agnostic about the positive role of TP^ in advancing further development of economic analyses of production.

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No potential conflict of interest has been reported by the authors.

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NOTES

1 This paper is naturally bound by single-product 'model(s)' of two inputs/factors, Labor (*L*) and Capital ('*K*'), which often serves convenient when utilizing certain software to construct the total product surface/hill in a 3-dimensional space. Nonetheless, if the concern is how difficult it seems to measure and aggregate units of different heterogeneous 'human-made' instruments, then perhaps *K* should be replaced in the first place with the great late Paul Samuelson's preferred choice—Land—as demonstrated in Chacholiades (1986, p.175); however, we would then be met with problems of farmland being dividable and/or fixed proportions persisting (more easily).

2 See Cramer (2010) for an introduction to the logistic growth curve. Meanwhile, we offer following notes in advance: the slanted and elongated S-shaped TP curve presented here is a quasi-concave function (Hands, 2004, p.46), and on the premise of it being (continuously) differentiable, this paper searched across past literature (see endnote 15 for further reference) for math functions that yield S-shaped TP curves and has decided to depict the MP and AP curves as (smooth and) inverted U-shaped. By contrast, some kind of first-order homogeneous function yielding an S-shaped TP curve—$Q=K^2L^2/(aK^3+bL^3)$ for example—had been developed separately by Ryuzo Sato and John W. Rowe Jr. back in 1964. But if we understand Truett and Truett (2006)—a text seldom quoted for unknown reasons, while still a must-read much like Intriligator (2002[1971], p.185)—correctly, the AP and MP curves of this functional form are of 'partial' (or degenerated?) bell-shape, in the sense that these curves might, depending on different parameters/coefficients, both have (at least) an inflection point *after leaving for the second stage* of production; besides, each of its isoquants would indicate both peculiar patterns of shape and unusual number of ridge lines, all of which (were probably discovered by Ragnar Frisch already before 1930 and) are lacking in the case of the Cobb-Douglas functions. We suspect that if these two curves are bell-shaped with inflection point(s), it would give rise to further problems that need addressing, which we leave to another occasion; and now, for narrating that (1) *some non-homogeneous functions do generate three stages of production and inverted U-shaped curves of AP and MP at the same time*, and (2) 'assuming first-order homogeneity' alone won't necessarily bring out S-shaped TP curves, this note would resolutely treat (figure 1b's) inverted U-shaped curves as distinct from the (Knightian-)Sato/Rowe type, denominated by Truett and Truett (2006, p.350), of bell-shaped AP and MP curves. Interested readers might check the second differential of $Q=-L^3+L^2+2L$—a non-homogeneous cubic polynomial soon to appear later—to ponder beforehand what we mean here: a homogeneous function like $Q=K^2L^2/(aK^3+bL^3)$ will not procure the inverted U-shaped AP and MP curves that have acted as the protagonists of almost all the textbooks for more than four decades, although it seems impossible to find the *global* inverse function for each of the exact functional forms.

3 The concept of variable/varying proportions is often taken as synonym for diminishing (marginal) returns/productivity; here, we contrast it against constant/fixed proportions. For pregnant discussion on increasing marginal returns, diminishing returns, and variable proportions, please see Zamagni (1987, ch.6)—a textbook that has long been overpassed.

4 From figure 1a to 1c, we have chosen not to plot any curve as connected to the origin point, in turn avoiding the problem of whether to apply, for example, **L'Hôpital's Rule**. One reason is that when textbooks start to discuss the three stages of production, rarely do they immediately touch on which mathematical functions create an S-shaped TP curve. Take our figure 1c as an example: it essentially maintains an *agnostic* view to the question of 'What is the length of its horizontal axis?' Secondly, after assuming the spatially vertical line as Q-axis and the two (orthogonally) horizontal ones as *L* and *K*, we utilized an online graphics software (GeoGebra) to examine the total product surfaces behind certain S-shaped TP curves (as listed in our endnote 15), and discovered that some of these surfaces resemble, visually, a long and narrow valley that extends northwest to southeast. An additional problem arises—when said '(compressed oval) valley-esque form' $Q=f(L,K)$ is continuously differentiable—if $0=f(1,0)=f(0,1)$ and Q-axis continues to slant downwards into the <0 area, then it often leads to (for example) $0> f(0,0)$ and $0> f(dL,dK)$, with dL and dK being within the positive epsilon neighborhood of (0,0). Thirdly, slightly advanced textbooks often choose to introduce the three stages in parallel with isoquants, such as mixing in oval-shaped isoquants and the ridge lines. Yet even textbooks that utilize software like Matlab to demonstrate economics never seem to touch on this: whether the elliptical cone–shaped total product surface can always sidestep the above problem.

5 We shall call the original three stages—namely figure 1b—the simple (version of) three stages. As for the question: Should stage one end at the highest point of the MP curve? Perhaps YES! But then there seem to be no theorems available for narrating and justifying—even for the wrong reason—this kind of symmetric three stages, which is probably why such classification of three stages has not been established. Furthermore, the extended version this paper discusses below is a general reference to figures with four curves that intersect at precise locations (namely figure. 1c), and within this expanded *family* of three stages there exists two minor versions, that is, there are two distinct pairs of variables on figure 1c's horizontal axis. One of them is *L and/or K*, which utilizes Euler's theorem and homogeneous functions of degree one; the other is *L and/or K/L*, which originates from the $TP\hat{}(L,K/L)$ discussed in this paper's penultimate section, so please note the literary context. In contrast to some literature, we refrain from thoroughly adopting the term symmetry of production stages because, according to Bilas

(1971, pp.133-138), when a homogeneous function is not of the first order, it does not bear symmetry; nevertheless, he still argues for other slightly varied extended versions, which might be depicted as *asymmetric* extended versions by our standard. However, if our reasoning in the penultimate section is accepted, then the arguments Bilas puts forth for figure 6-12 and 6-13 in his book—along with Mundlak's (1958, p.758) '[table of] classification of stages of production'—should not be taken seriously. In other words, the *asymmetric* extended version would also be implausible.

6 Whaling involves natural resources and common ownership externality; the two-input explanation might not be enough for its real production pattern, nor could 'free disposal'(Puu, 2018, p.18) lead to straightforward elimination of the third stage.

7 In Friedman (1962, pp.122-131), Hibdon (1969, pp.119-124) and Holland (1973, pp.133-134), because they used some kind of input-output schedules comprised of discrete data to set up the extended three stages, K=1 (or L=1) naturally benefits calculation at first glance; see also Truett and Truett's (2006, p.351) K=10.

8 We acknowledge frankly that: even if the mathematical deduction offered in our penultimate section is flawless, it still does not guarantee that any production function (rewritten) with K/L as an/the independent variable will accurately yield its own $AP_{K/L}$ and $MP_{K/L}$ functions; after all, a (homogeneous!) function like $Q=(K/L)^{-1}$ cannot be reformulated into a TP^ function as shown in the penultimate section. To those who presume any extended version to be already mathematically impeccable should still consider the problem of whether the horizontal axis in figure 1c has a definitive length.

9 Unfortunately, due to the limited library available at our academic institution, we were unable to locate these textbooks with different editions *and* verify whether their authors ever applied Euler's theorem such way in some earlier editions/works. Additionally, we recommend heeding Bilas' figure 6-11-b as well as Doll and Orazem's figure 6.6, although they might have adopted variable symbols different from figure 1c here.

10 In this paragraph, we have twice specifically established '*if both curves are inverted U-shaped*' as an indispensable but often neglected premise for both formula (2) and (3), because the lure of manipulating mathematical formulas or axioms often hinders whatever helpfulness a figure might have served us: not every homogeneous function of degree one can generate three stages of production. We shall soon see hereinafter: given that formula (2) and (3) both include *three* related statements separated by MP=0, it makes them seemingly suitable for explaining the extended *three* stages; however, the triad statements might have somehow distorted a characteristic of some first-order homogeneous function. If researchers could, when applying Euler's theorem, also *meticulously tend to* its objects' various possible figures—our figure 2b for example—then this paper would not exist.

11 This paper, to repeat, focuses on *S-shaped TP curve* (with its inverted U-shaped AP and MP curves as a whole), which will never take the form of equations like $Q=aL+aK$ (a>0)—where there is only one MP curve with two names; and we would also not consider equations in the form of $Q=a_1L+a_2K$ ($a_1 \neq a_2$, a_1>0, a_2>0) to be the (sole) standard representation of a first-order homogeneous function. Furthermore, the simultaneous presence of four curves in figure 1c suggests strongly that: on a certain total product surface in three-dimensional space, K=1 for example was used to obtain a cut plane (namely some TP_L curve), and then another cut plane was derived with L set as fixed amount (namely a TP_K curve). Aside from questioning whether it is appropriate to combine cut planes in such 'overlapping' manner, in the ensuing text about to cope with a cubic polynomial, we would swiftly inquire: which of the latter cut planes should be chosen?

12 On the contrary, why any first-order homogeneous function seems, at first glance, able to be exempt from such illogicality? We leave it to the final section.

13 Please look back to the earlier endnote 2.

14 Panik (2022, p.226) mentions 'Euler's theorem for almost homogeneous functions,' but that doesn't seem to come to the rescue for our subject.

15 The letters representing variables have been modified (the same goes for subsequent text), and the following equations make it even easier to see that all of them are non-homogeneous functions: $Q=-0.5L^3+7.5L^2$ (Keating & Keating, 2009, p.189), $Q=-L^3+aL^2$ (Otani & El-Hodiri, 1987, p.72), $Q=-a_1L^3+a_1a_2L^2$ (Puu, 1985, p.1264), $Q=-0.0012L^3K^3+3.5L^2+4.6K^2$(Rosser & Lis, 2016, p.330), $Q=-a_1L^3+a_2L^2K^2$, a_1>0, a_2>0 (Shone, 1975, p.188). Here we have tried our best in being faithful to the above publications when simplifying or rewriting them, yet we are unable to detail each author's consideration when it comes to equations'

different coefficients and the reasons for given K's values—considerations that are partly related to the earlier endnote 4. For instance, every TP curve is a cut plane of a three-dimensional total product surface, so the different coefficients may not always connect the former to the origin point of the plane; this seems to be a 'comparative disadvantage' in contrast to $Q=K^2L^2/(aK^3+bL^3)$.

16 Once again, although homogeneous functions as such don't have generalized formulas like Cobb-Douglas functions do, if the latter—as accredited candidate for application of Euler's theorem—cannot generate S-shaped TP curves, then the argument laid out so far should be regarded as even more convincing; nevertheless, if any textbook author prefers the Sato/Rowe type, then the latter's isoquants should be acknowledged and stated, sooner or later, clearly to have unusual shape and number of ridge lines (Truett and Truett, 2006, p.353) than we might have expected. Meanwhile, when the homogeneous function is not of degree one—in which case the formula (1) must be modified—would its TP curve be unexpectedly S-shaped? We leave this question to our readers (perhaps with aid from GeoGebra). It is worth mentioning that in Clower et al. (1988, p.141, note 3, in particular), they first established a Cobb-Douglas function—$Q=L^{\alpha}K^{1-\alpha}$ (we have modified the variable's lettering)—before stating that 'for [*L*'s first stage of production] to hold, α must be greater than 1.' Yet, if that were the case, then not only MP_L curves still won't be inverted U-shaped, all the MP_K values of the same production function would also be negative throughout, and therefore the second and third stages in their figure 7-2 will be radically nonexistent!

17 At this point, we see how the first half of note 7 in Tangri (1966, p.493) was inappropriate, because not every first-order homogeneous function will have $MP \leq 0$. Besides, referring back to formula (1), it also could be divided by Q to obtain something more concise, via employing the concept of output elasticities of *L* and *K* respectively, than formula (2) and (3); here, it cannot be overstated that *both of these elasticities of the Cobb-Douglas are inherently positive and each less than 1*!

18 In case some readers may not be patient enough to follow our reasoning this far, let us present a perhaps somewhat cliché and unnecessary test on formula (2)—while formula (3) left to our readers—but one that may spare us from the criticism of 'relying utterly on graphs.' Once again, we turn to the example of $Q=-L^3-K^3+a_1a_2L^2+a_1a_2K^2+2a_1a_2LK$; on one hand, $AP_L=-L^2-K^3(1/L)+a_1a_2L+a_1a_2K^2(1/L)+2a_1a_2K$, but on the other, $MP_L+MP_K\cdot(K)(1/L)= -3L^2-3K^3(1/L)+2a_1a_2L+2a_1a_2K^2(1/L)+4a_1a_2K$. After all, formula (2) and (3) originate from formula (1) and, as mentioned earlier, this cubic polynomial as such does not possess the characteristic of $Q=MP_L\cdot L+MP_K\cdot K$ inherently.

19 Be sure to take notice of this: '$\hat{MP}_L(K/L)=\partial \hat{TP}(L,K/L)/\partial L_{(DE)}$' here is defined as *L*'s *direct effect* on $\hat{TP}(L,K/L)$; and as the following steps B and C will show more thoroughly, *L*'s *indirect* effect is $[\partial \hat{TP}(L,K/L)/\partial(K/L)]\cdot[d(K/L)/dL]$.

20 Please feel free and welcome to request the first author among us for it.